Topology of Representation Spaces via Arithmetic

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Representation Spaces

Why do we care about representation spaces?

In mathematics and physics, several closely related moduli spaces arise:

Space of Higgs bundles	$\mathcal{M}_{\text{Dolbeault}}$	$\mathcal{M}_{\text{deRham}}$	Space of flat connections on a vector bundle
Space of solutions to Hitchin's equations	$\mathcal{M}_{\text{Hitchin}}$	$\mathcal{M}_{\text{Betti}}$	Space of representations of a fundamental group

$$\begin{array}{ccc} \mathcal{M}_{\text{Dolbeault}} & & \mathcal{M}_{\text{deRham}} \\ & \searrow & \updownarrow \\ \mathcal{M}_{\text{Hitchin}} & \longleftrightarrow & \mathcal{M}_{\text{Betti}} \end{array}$$

These spaces are central to many abstract and physical topics, including Yang-Mills theory, Hitchin's equations, the Langlands program, the $P = W$ conjecture, and mirror symmetry.

What are representation spaces?

Fix a surface with fundamental group Γ, an algebraic group G and a conjugacy class $C \subseteq G$. In this project, we consider an orientable surface with genus $g \geq 1$ and one puncture:

$$\Gamma := \pi_1\Big(\qquad\Big) \simeq \frac{\langle x_1, y_1, \ldots, x_g, y_g, z\rangle}{[x_1, y_1]\cdots[x_g, y_g]z}$$

We are interested in two representation spaces:

- ⋆ The representation variety $R := \{f \in \operatorname{Hom}(\Gamma, G) \mid f(z) \in C\}$
- ⋆ The character variety $X := R /\!\!/ G$

The quotient $R/\!\!/G$ above is the Geometric Invariant Theory (GIT) quotient. We can form this quotient because there is an action $G \curvearrowright R$ given by conjugation.

Topology

E-polynomials

To study the representation spaces, we compute their E-polynomials. This is a specialisation of the Hodge polynomial which encodes fine cohomological data.

$$\text{Hodge polynomial } H(q,t) \xrightarrow{t=-1} E\text{-polynomial } E(q), \qquad E(q) \overset{??}{\dashrightarrow} H(q,t)$$

It is extremely difficult to compute Hodge polynomials. However, computing the specialisation $H(q, -1) = E(q)$ has guided conjectural formulas for Hodge polynomials and aided full proofs.

Topological information

If Y has E-polynomial $E(q)$ then we can access the following topological information:

The degree of $E(q)$ is the dimension of Y

The value $E(1)$ is the Euler characteristic of Y

The leading coefficient of $E(q)$ is the number of components of Y

For instance,

$$Y = GL_3\text{-flag variety} \quad \rightsquigarrow \quad |Y(\mathbb{F}_q)| = |GL_3(\mathbb{F}_q)|/|B(\mathbb{F}_q)| = q^3 + 2q^2 + 2q + 1$$

Therefore Y has dimension 3, Euler characteristic 6 and is connected.

Arithmetic

Weil's conjectures & Katz' theorem

To compute E-polynomials, we rely on the Weil conjectures, a jewel of 20th century mathematics. The conjectures (now theorems) are technical, but they teach us an important philosophy:

Cohomological information can be obtained by counting points over finite fields

A theorem due to Katz' refines this philosophy:

Theorem 1 (Katz). *Suppose that Y is a variety and $|Y(\mathbb{F}_q)|$ is given by some polynomial $P(q)$. Then the E-polynomial of Y is given by*

$$E(q) = P(q).$$

Character sums are polynomial

Once Problem ② is solved, the polynomiality of $|X(\mathbb{F}_q)|$ reduces to the following problem:

Problem 2. Suppose that $T \subseteq G$ is a split maximal torus and that $S \in T(\mathbb{F}_q)$. Moreover, fix a closed root subsystem $\Psi \subseteq \Phi^\vee$.

Show that the 'character sum' defined by

$$\sum_{\substack{\theta \in \operatorname{Irr}(T(\mathbb{F}_q)) \\ W_\theta = W(\Psi)}} \theta(S)$$

is a polynomial in q.

This problem was solved in [KNP], where it was concluded that it is 'essentially' polynomial.

Frobenius' mass formula

Frobenius tells us how to count $\mathbb{F}_q$-points of the representation variety.

$$\frac{|R(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \sum_{\chi \in \operatorname{Irr}(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)}\right)^{2g-2} \frac{\chi(C(\mathbb{F}_q))}{\chi(1)} |C(\mathbb{F}_q)|$$

We want to show that this is a polynomial in q and compute its features.

Problems:

1. $\operatorname{Irr}(G(\mathbb{F}_q))$ is unknown in general
2. $\operatorname{Irr}(G(\mathbb{F}_q))$ depends on q
3. $\chi(C(\mathbb{F}_q))$ is unknown in general

Solutions:

1. Choose G reductive so that we know $\operatorname{Irr}(G(\mathbb{F}_q))$
2. Write sum over data independent of q
3. Choose C 'semisimple, regular generic' so that we know $\chi(C(\mathbb{F}_q))$

Literature

In [HRV, HLRV], the authors considered $G = \mathrm{GL}_n$ with k 'semisimple generic' conjugacy classes, with $\mathcal{N}_C$ recording the multiplicities of their eigenvalues.

Theorem 4. *The $\mathbb{F}_q$-points of the character variety X is polynomial in q and*

- ⋆ *The dimension of X is $(2g - 2 + k)n^2 - \mathcal{N}_C + 2$*
- ⋆ *The Euler characteristic is almost always 0*
- ⋆ *X is connected*
- ⋆ *The coefficients of $|X(\mathbb{F}_q)|$ are a palindrome*

Our main theorem agrees with these results.

Results

Theorem 3 (Kamgarpour-Nam-W. 2023). *Let G be a connected split reductive group with connected centre Z. Let $C \subseteq G$ be a 'semisimple regular generic' conjugacy class.*

Then the $\mathbb{F}_q$-points of the character variety X is polynomial in q and

- ⋆ *The dimension of X is $(2g - 1)\dim G - \operatorname{rank} G + 2\dim Z$*
- ⋆ *The Euler characteristic of X is 0 if $g > 1$ or $\dim Z > 0$*
- ⋆ *The number of components of X and the centre of $G^\vee$ are the same*
- ⋆ *The coefficients of $|X(\mathbb{F}_q)|$ are a palindrome*
  This points towards a 'curious' Poincare duality

Literature

In [Cambò], the author considered $G = \mathrm{Sp}_{2n}$ and a 'semisimple regular generic' conjugacy class.

Theorem 5. *The $\mathbb{F}_q$-points of the character variety X is polynomial in q and*

- ⋆ *The dimension of X is $(2g - 1)n(2n + 1) - n$*
- ⋆ *The Euler characteristic is almost always 0*
- ⋆ *X is connected*
- ⋆ *The coefficients of $|X(\mathbb{F}_q)|$ are a palindrome*

Despite the centre of Sp_{2n} being disconnected, the results are strikingly similar.

Visualisations

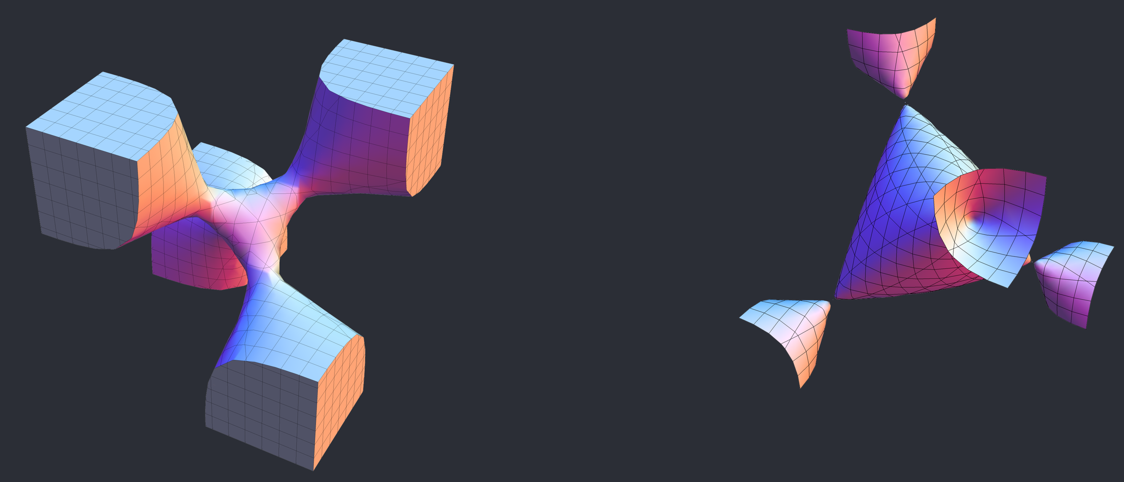

Figure 1: When $G = \mathrm{SL}_2$ and $\Gamma = \pi_1(\text{Torus}) \simeq \langle x, y \mid xy = yx\rangle$, we obtain the Cayley cubic. The Cayley cubic's defining equation is $16xyz + 12(x^2 + y^2 + z^2) = 27$.

Loose ends and open problems

- ⋆ What happens for different conjugacy classes? What if the surface has multiple punctures?
- ⋆ What is the mixed Hodge polynomial of these representation spaces?
- ⋆ When $G = \mathrm{GL}_n$, there is a strong combinatorial theory. In [HRV, HLRV], the authors used symmetric functions and Macdonald polynomials. Can we count points of representation spaces using these combinatorial ideas as well?
- ⋆ When $g = 1$, the Euler characteristic E_n of an Sp_{2n}-character variety was given in [Cambò]:

$$\sum_{n \geq 0} \frac{E_n}{2^n n!} T^n = \prod_{k \geq 1} \frac{1}{(1 - T^k)^3} = 1 + 3T + 9T^2 + \cdots.$$

Can we obtain an expression for the Euler characteristic when $g = 1$ and $\dim Z = 0$?

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