Arithmetic Geometry of Representation Spaces

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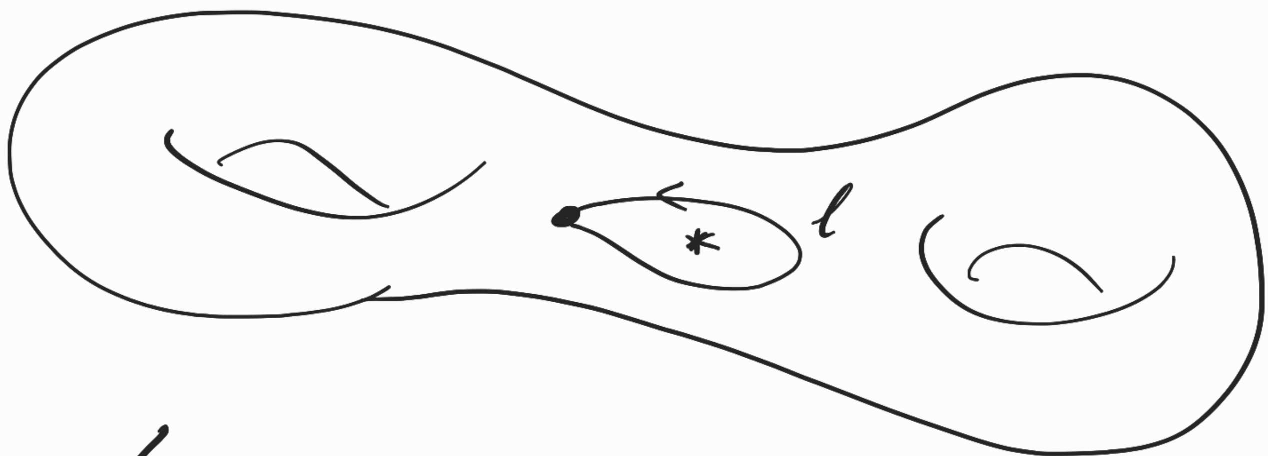
Representation Spaces

- G = algebraic group $/\mathbb{F}_q$ $\quad GL_n, SL_n, PGL_n, Sp_{2n}, \ldots$
- $\Gamma = \pi_1$(orientable surface genus $g \geq 1$, one puncture)



$$\Gamma \cong \left\langle x_1, y_1, \ldots, x_g, y_g, l \;\middle|\; [x_1,y_1]\ldots[x_g,y_g]\,l = 1 \right\rangle.$$

- $C \subseteq G$ conj. class

<u>Def</u> The representation variety is

$$R := \{ f \in \mathrm{Hom}(\Gamma, G) \mid f(\ell) \in C \}.$$

$$\simeq \left\{ (A_1, \dots, B_g, L) \in G^{2g} \times C \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] L = 1 \right\}$$

$G \curvearrowright R$ by conjugation:

- $(g \cdot f)(x) := g f(x) g^{-1}$
- $g \cdot (A_1, \dots, B_g, L) := (g A_1 g^{-1}, \dots, g L g^{-1}).$

The character variety is

$$X = R /\!\!/ G.$$

Many connections:

X $\quad M_{\text{Betti}}$.

Space of Higgs bundles M_{Dol}

Space of solutions to Hitchin's equations M_{Hit}

Space of flat connections M_{dR}

<u>Goal</u>: Study the character variety.

<u>How to study the char var?</u>

Y = variety over $\mathbb{Z}$

<u>Moral [Weil]</u>: Count $|Y(\mathbb{F}_q)|$ $\rightsquigarrow$ Understand $H^*(Y)$

<u>Theorem [Katz]</u>: If $q \longmapsto |Y(\mathbb{F}_q)|$ is poly in q then this poly encodes info about $H^*(Y)$.

eg. $Y = GL_3/B$, $|Y(\mathbb{F}_q)| = q^3 + 2q^2 + 2q + 1$.

$\dim Y$ = degree of poly = 3

$\chi(Y)$ = poly at $q=1$ = 6

# of components of ~~maximum dim~~ = leading coeff = 1.

Y is smooth

How to count $\mathbb{F}_q$-points?

Frobenius:

$$\frac{|R(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-2} \frac{\chi(C(\mathbb{F}_q))}{\chi(1)} |C(\mathbb{F}_q)|.$$

Goal: Show poly in q.

Problems:

1. $\mathrm{Irr}(G(\mathbb{F}_q))$ is hard e.g. $\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}\right\}$.
2. $\mathrm{Irr}(G(\mathbb{F}_q))$ depends on q.
3. $\chi(C(\mathbb{F}_q))$ is difficult to evaluate.

Solutions:

1. G reductive $\rightsquigarrow$ Deligne-Lusztig theory.
2. Massage dependence on q.

$$\mathrm{Irr}(G(\mathbb{F}_q)) \ni \chi \longmapsto \text{Data indep of } q.$$

"$\chi(1)$" $\leftsquigarrow$ reconstruct the degree.

③ C semisimple regular & generic.

<u>$G = GL_n$</u>: $C = \left[\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \right]$

diagonal $\Longrightarrow$ semisimple

$x_i \neq x_j \Longrightarrow$ regular

$\prod_i x_i = 1$ & no subproduct $= 1$ $\overset{GL_n}{\Longleftrightarrow}$ generic

<u>2008 - 2011</u>: Hausel - Letellier - Rodriguez - Villegas.

$G = GL_n$, many puncs., $C_1, \dots, C_k$
semisimple & generic.

$X = R /\!/ GL_n$.

Theorem [HLRV '11]: $|X(\mathbb{F}_q)|$ poly in q.

- $\dim X = (2g-2+k)n^2 - 2 + N_c$
- $\chi(X) = 0$ if $g \geq 1$.
- X smooth & connected.
- $|X(\mathbb{F}_q)|$ palindromic

  $\hookrightarrow$ "Poincare duality"

2017: Vincenzo Cambò

$G = Sp_{2n}$, one punc. ss reg generic.

$X_n = R /\!/ Sp_{2n}$

Theorem [Cambo]: $|X_n(\mathbb{F}_q)|$ poly in q.

- $\dim X_n = (2g-1)n(2n+1) - n$.
- $\chi(X_n) = 0$ if $g > 1$ & if $g = 1$ then

$$\sum_{n \geq 0} \frac{\chi(X_n)}{n!\, 2^n} T^n = \prod_{k \geq 1} \frac{1}{(1-T^k)^3} = 1 + 3T + 9T^2 + \ldots$$

- X smooth & connected.
- $|X(\mathbb{F}_q)|$ palindromic too.

<u>2023:</u> Kamgarpour - Nam - W.

G = conn. split red gp w/ conn centre.

C = ss reg generic. ← Type indep!

$X = R /\!/ G$.

<u>Theorem [KNW]:</u> $|X(\mathbb{F}_q)|$ is poly.

- $\dim X = (2g-1)\dim G - 2\dim Z + \operatorname{rank} G$.
- $\chi(X) = 0$ if $g > 1$ or $\dim Z > 0$
- X is smooth but not necessarily connected.

# of components of X

=

# of components of $Z(G^\vee)$.

$= 1$ if $G = GL_n$ or Sp_{2n}.

- $|X(\mathbb{F}_q)|$ palindromic

<u>Techniques:</u>

<u>Moral [Langland]:</u> $Irr(G(\mathbb{F}_q))$ is controlled by $G^\vee(\mathbb{F}_q)$.

<u>Theorem [Lusztig]:</u> If G has conn centre then

$$Irr(G(\mathbb{F}_q)) \longleftrightarrow \bigsqcup_{\substack{[x] \subseteq G^\vee(\mathbb{F}_q) \\ \text{ss. conj.}}} Uch(G^\vee_x(\mathbb{F}_q))$$

- $G^\vee_x(\mathbb{F}_q)$ = centraliser of x in $G^\vee(\mathbb{F}_q)$.
- $Uch(G^\vee_x(\mathbb{F}_q)) \subseteq Irr(G^\vee_x(\mathbb{F}_q))$

independent of q.

depends on q.

eg. $$\mathrm{Uch}(GL_n(\mathbb{F}_q)) \longleftrightarrow \left\{\begin{array}{c}\text{partitions}\\ \text{of } n\end{array}\right\}.$$

$$\chi \longmapsto ([x], \rho) \longmapsto (\Psi, \omega, \rho)$$

dep on q.

indep of q

(Ψ "$\chi(1)$")

[BK22].

<u>Carter:</u> $[x] \longmapsto (\Psi, \omega)$

Ψ = root sys of $G_x^\vee(\mathbb{F}_q) \subseteq \Phi^\vee$.

$$\omega \in N_W(W(\Psi))/W(\Psi).$$

This reduces the calculation of $|X(\mathbb{F}_q)|$ to the following problem:

Take
- $T \subseteq G$ split maximal torus
- $S \in T(\mathbb{F}_q)$.

- $\Psi \subseteq \Phi^\vee$ closed subsys.

Show that

$$\alpha_{\Psi,s}(q) := \sum_{\substack{\theta \in T(\mathbb{F}_q)^\vee \\ \mathrm{stab}_W(\theta) = W(\Psi)}} \theta(s)$$

is polynomial in q.

This was solved in [KNP].

Anna Puskás