The Deligne–Simpson problem, braid stacks, and computers

Bailey Whitbread

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The shape of our theorem

Simple gp. $G/\mathbb{C}$ Conj. class C in G Rational number ν	$\rightsquigarrow$	Is there a 'G-conn.' on $\mathbb{P}^1_{\mathbb{C}}$ with 'monodromy' C at ∞ and 'slope' ν at 0?
(G, C, ν)	$\mapsto$	yes/no

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Theorem (Kamgarpour–W., '26)
If G is exceptional, check our table

Theorem (Jakob–Yun, '23)
If G is classical, check our table

An old problem

Abstract gp. G and conj. classes $C_1, \ldots, C_r$ in G

Problem: Do there exist $g_i \in C_i$ with $g_1 \cdots g_r = 1$?

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Abstract gp. G and conj. classes $C_1, \dots, C_r$ in G

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If G is finite, the number of such tuples $(g_1, \dots, g_r)$ is

$$\frac{|C_1| \cdots |C_r|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(C_1) \cdots \chi(C_r)}{\chi(1)^{r-2}}$$

An old problem

53.

Über Gruppencharaktere

Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin
985 – 1021 (1896)

p, ℓ prime

$px + \ell \quad x = 1, 2, 3, \ldots$

Bei dem Beweise des Satzes, dass jede lineare Function einer Variabeln unendlich viele Primzahlen darstellt, wenn ihre Coefficienten theilerfremde ganze Zahlen sind, benutzte Dirichlet zum ersten Male gewisse Systeme von Einheitswurzeln, die auch in der nahe verwandten Frage nach der Anzahl der Idealclassen in einem Kreiskörper auftreten (vergl. die Bemerkung von Dedekind in Dirichlet's Vorlesungen über Zahlentheorie, 4. Aufl. S. 625), sowie bei der Verallgemeinerung jenes Satzes auf quadratische Formen und in den Untersuchungen über deren Eintheilung in Geschlechter. Die charakteristische Eigenschaft dieser Ausdrücke besteht nach Dedekind darin, dass sie von einer variabeln positiven ganzen Zahl n abhängige Grössen $\chi(n)$ sind, die nur eine endliche Anzahl von Werthen haben und der Bedingung

$$\chi(m)\chi(n) = \chi(mn)$$

genügen. Wie er in rein abstracter Form ausführt, lassen sich den Elementen $A, B, C, \cdots$ jeder endlichen Gruppe $\mathfrak{H}$ vertauschbarer Elemente (Abel'schen Gruppe) solche Einheitswurzeln $\chi(A), \chi(B), \chi(C), \cdots$ zuordnen, welche die Gleichungen

$$\chi(A)\chi(B) = \chi(AB)$$

befriedigen, und die er nach dem Vorgange von Gauss *die Charaktere der Gruppe* nannte.

Seien (α), (β), (γ) irgend drei verschiedene oder gleiche Classen. Durchläuft A die h_α verschiedenen Elemente der α^{ten} Classe, B die h_β Elemente der β^{ten} und C die h_γ Elemente der γ^{ten}, so soll die Zahl $h_{\alpha\beta\gamma}$ (die auch Null sein kann) angeben, wie viele der $h_\alpha h_\beta h_\gamma$ Elemente ABC gleich dem Hauptelemente sind, also der Gleichung

$$ABC = E \tag{3.}$$

genügen. Da dann $AB = C^{-1}$ ist, so giebt $h_{\alpha\beta\gamma}$ auch an, wie viele der $h_\alpha h_\beta$ Elemente AB der Classe (γ') angehören. Die Gleichung (3.) ist identisch mit $BCA = E$ und $CAB = E$. Daher sind auch $h_{\alpha\beta\gamma}$ der $h_\beta h_\gamma$ Producte BC in (α') und $h_{\alpha\beta\gamma}$ der $h_\gamma h_\alpha$ Producte CA in (β') enthalten. Mithin ist $h_{\alpha\beta\gamma}$ nicht grösser als die kleinste der drei Zahlen $h_\beta h_\gamma$, $h_\gamma h_\alpha$ und $h_\alpha h_\beta$.

An old problem

Suppose $G = \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C}), G_2(\mathbb{C}), F_4(\mathbb{C}), \ldots$

This is called the 'Deligne–Simpson problem'

Products of Matrices

CARLOS T. SIMPSON

Introduction

We will consider the matrix equation

$$A_1 \cdots A_k = 1$$

for matrices A_i in specified conjugacy classes $C_i \subset SL(n, \mathbb{C})$. The question is whether there exists a solution.

THEOREM. *Suppose C_k is semisimple with generic distinct eigenvalues. Then there exists a solution $A_1 \cdots A_k = 1$ if and only if $\sum \dim(C_i) \geq 2n^2 - 2$ and $r_1 + \cdots + r_{k-1} \geq n$.*

An old problem

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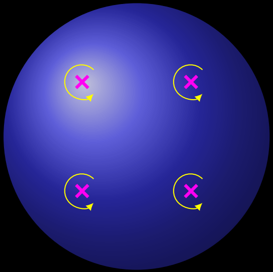
COUNTING POINTS ON GENERIC CHARACTER VARIETIES

STEFANO GIANNINI[1], MASOUD KAMGARPOUR[2], GYEONGHYEON NAM[3], AND BAILEY WHITBREAD[4]

An old geometric problem

$X = \mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \ldots, p_r\}$

$\pi_1(X) = \langle \ell_1, \ldots, \ell_r \ : \ \ell_1 \cdots \ell_r = 1 \rangle$



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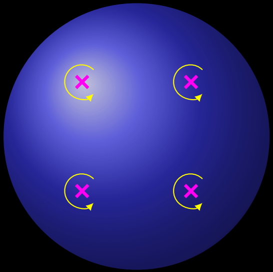

Tuples $(A_1, \ldots, A_r)$ in $\mathrm{GL}_n(\mathbb{C})$ with $A_1 \cdots A_r = 1$ and $A_i \in C_i$

$\Updownarrow$

Local systems on X with monodromy C_i around p_i

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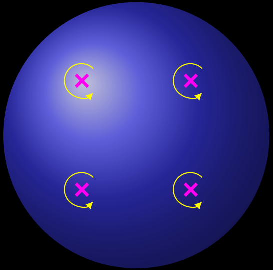

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with $A_1 \cdots A_r = 1$ and $A_i \in C_i$

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GL_n-connections on X
with reg. sing. at p_i
and monodromy in C_i

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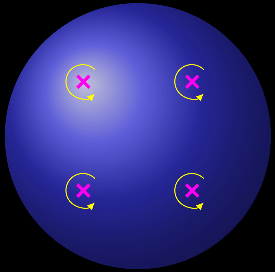

$$\nabla = d + B(t)\,dt$$

$$B(t) \in \mathfrak{gl}_n((t))$$

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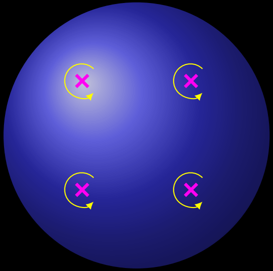

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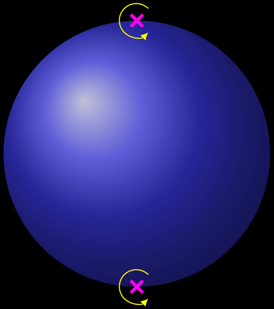
Tuples $(A_1, \ldots, A_r)$ in $G(\mathbb{C})$
with $A_1 \cdots A_r = 1$ and $A_i \in C_i$

$\Updownarrow$

G-connections on X
with reg. sing. at p_i
and monodromy in C_i

A modern geometric problem

$$X = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$$



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$\rightsquigarrow$ / $\leftrightsquigarrow$

G-connections on X
with reg. sing. at ∞
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A modern geometric problem

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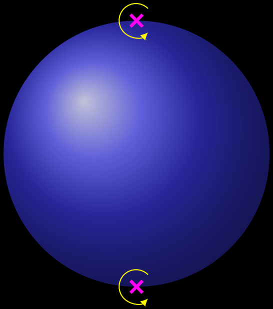

$\nabla = d + B(t)\, dt$

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Complex points on the stack $\mathcal{M}(C, \beta)$

$\updownarrow$

G-connections on X with reg. sing. at ∞ and irreg. sing. at 0

The braid stack

Conj. class C in G

Positive braid $\beta = \widetilde{w_1} \cdots \widetilde{w_r}$ for G

The braid stack is

$$\mathcal{M}(C, \beta) = \left\{ (g, F_1, \ldots, F_{r+1}) \ : \ \begin{array}{c} g \in C, \ F_i \in G/B \\ F_i \xrightarrow{w_i} F_{i+1}, \ gF_{r+1} = F_1 \end{array} \right\} \Big/ G$$

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G-connections on X
with reg. sing. at ∞
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$\leftrightsquigarrow$

Complex points on
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Determining the braid

G-connections on X
with reg. sing. at ∞
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Determining the braid

G-connections on X with reg. sing. at ∞ and irreg. sing. at 0 $\leftrightsquigarrow$ Complex points on the stack $\mathcal{M}(C, \beta)$

Assumption: Around 0 the conn. mat. $B(t) \in \mathfrak{g}((t))$ has regular semisimple leading term

Determining the braid

G-connections on X with reg. sing. at ∞ and irreg. sing. at 0 $\longleftrightarrow$ Complex points on the stack $\mathcal{M}(C, \beta)$

Assumption: Around 0 the conn. mat. $B(t) \in \mathfrak{g}((t))$ has regular semisimple leading term

Consequence: β is explicitly computable using the 'slope' ν of $B(t)$

$$\nu = d/m \rightsquigarrow \beta \text{ `$=$' } \sqrt[m]{\Pi}^{d}$$

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Conj. class C in G	$\rightsquigarrow$	with 'monodromy' C at ∞
Rational number ν		and 'slope' ν at 0?
(G, C, ν)	$\mapsto$	yes/no

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THE SHAPE OF OUR THEOREM

Simple gp. $G/\mathbb{C}$ Conj. class C in G Rational number ν	$\rightsquigarrow$	Is there a complex point on the braid stack $\mathcal{M}(C, \sqrt[m]{\Pi}^{d})$?
(G, C, ν)	$\mapsto$	yes/no

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Aside: Lusztig's map

Connected reductive gp. G — Weyl group W

Unipotent classes $UC(G)$ — Conj. classes $CC(W)$

In 2011, Lusztig constructed a 'miraculous' surjective map

$$CC(W) \to UC(G)$$

$$[w] \mapsto \text{min. class intersecting } BwB$$

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Proof for exceptional G: Count points on $C \cap BwB$ over $\mathbb{F}_q$

Tight relationship: $\mathcal{M}(C, \beta) \simeq (C \cap BwB)/B$ when $\beta = \widetilde{w}$

The shape of our proof

Lusztig's formula for counting points on $C \cap BwB$ over $\mathbb{F}_q$ is

$$|(C \cap BwB)(\mathbb{F}_q)| = \frac{1}{|G(\mathbb{F}_q)|} \sum_{g \in C(\mathbb{F}_q)} \sum_{E \in \mathrm{Irr}(W)} \mathrm{tr}(g, U_E)\,\mathrm{tr}(T_w, H_E)$$

where

U_E is a unipot. $G(\mathbb{F}_q)$-rep. & H_E is an irred. Hecke alg. rep.

The shape of our proof

Our formula for counting points on $\mathcal{M}(C, \beta)$ over $\mathbb{F}_q$ is

$$|\mathcal{M}(C,\beta)(\mathbb{F}_q)| = \frac{1}{|G(\mathbb{F}_q)|} \sum_{g \in C(\mathbb{F}_q)} \sum_{E \in \mathrm{Irr}(W)} \mathrm{tr}(g, U_E)\,\mathrm{tr}(T_\beta, H_E)$$

where

$$\beta = \widetilde{w_1} \cdots \widetilde{w_r} \qquad \rightsquigarrow \qquad T_\beta = T_{w_1} \cdots T_{w_r}$$

Julia

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   _       _ _(_)_     |  Documentation: https://docs.julialang.org
  (_)     | (_) (_)    |
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Masoud 12:55

Let us call a positive braid $b\in B_W^+$ quasi-periodic if there exists a positive integer d and **even** integers $d_1,\dots,d_n$, and parabolic subgroups $W_1\supset W_2 \dots\supset W_r$ such that

$b^d = \underline{w_1}^{d_1} \underline{w_2}^{d_2} \dots \underline{w_r}^{d_r}$

Here, w_r is the longest element of W_r.

Question: Is every quasi-periodic braid nice?

Bailey 15:33

If we allow non-standard parabolics, then the conjecture quasi-periodic -> nice is false. It took Codex 3 hours to find this counter-example! (It gave up after 1.5 hours and I had to encourage it...)

Let W_1 = <2,3,4> and W_2 = x <1,3> x^{-1} with x = 213234 in W(F4). The former is a superset of the latter because x * 1 * x^{-1} = 323 and x * 3 * x^{-1} = 23432. Now take β = 2324323 to be the not-nice braid in F4 in our paper. Then β^6 = (w_1)^2 * (w_2)^4 where w_1 = 232343234 and w_2 = 232432 = x * 13 * x^{-1}.

Bailey 15:45

There is a counter example for standard parabolics too. Let $W_1 = \langle 2,3,4 \rangle$ and $W_2 = \langle 3,4 \rangle$. Then $\underline{\beta}^6 = (\underline{w_3})^6 * (\underline{w_4})^{(-4)}$ where $w_3 = 232343234$ and $w_4 = 343$.

Masoud 13:01

I just ran a codex experiment and it seemed to tell me that in S_n (for n\leq 9), every two m-Springer elements are related by a sequence of cyclic shifts...

Bailey 14:43

I've double checked that final calculation using Codex. To be specific, I checked, after fixing a regular number m, that all m-Springer elements are related under Geck-Pfeiffer's cyclic shift moves. I checked this for the regular non-elliptic numbers in E6 and E7, and also a sanity check for all regular numbers for G2 and F4.

So that completes our justification for using the specific m-Springer elements found in Broue-Michel to generate the tables in the appendix.