A Deligne–Simpson problem

Bailey Whitbread

University of Sydney

An old problem

Abstract gp. G and conj. classes $C_1, \ldots, C_r$ in G

Problem: Do there exist $g_i \in C_i$ with $g_1 \cdots g_r = 1$?

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If G is finite, the number of such tuples $(g_1, \ldots, g_r)$ is

$$\frac{|C_1| \cdots |C_r|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(C_1) \cdots \chi(C_r)}{\chi(1)^{r-2}}$$

An old problem

53.

Über Gruppencharaktere

Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin
985–1021 (1896)

P, ℓ prime

$px+\ell \quad x = 1, 2, 3, \ldots$

Bei dem Beweise des Satzes, dass jede lineare Function einer Variabeln unendlich viele Primzahlen darstellt, wenn ihre Coefficienten theilerfremde ganze Zahlen sind, benutzte Dirichlet zum ersten Male gewisse Systeme von Einheitswurzeln, die auch in der nahe verwandten Frage nach der Anzahl der Idealclassen in einem Kreiskörper auftreten (vergl. die Bemerkung von Dedekind in Dirichlet's Vorlesungen über Zahlentheorie, 4. Aufl. S. 625), sowie bei der Verallgemeinerung jenes Satzes auf quadratische Formen und in den Untersuchungen über deren Eintheilung in Geschlechter. Die charakteristische Eigenschaft dieser Ausdrücke besteht nach Dedekind darin, dass sie von einer variabeln positiven ganzen Zahl n abhängige Grössen $\chi(n)$ sind, die nur eine endliche Anzahl von Werthen haben und der Bedingung

$$\chi(m)\chi(n) = \chi(mn)$$

genügen. Wie er in rein abstracter Form ausführt, lassen sich den Elementen $A, B, C, \cdots$ jeder endlichen Gruppe $\mathfrak{H}$ vertauschbarer Elemente (Abel'schen Gruppe) solche Einheitswurzeln $\chi(A), \chi(B), \chi(C), \cdots$ zuordnen, welche die Gleichungen

$$\chi(A)\chi(B) = \chi(AB)$$

befriedigen, und die er nach dem Vorgange von Gauss die *Charaktere der Gruppe* nannte.

Seien (α), (β), (γ) irgend drei verschiedene oder gleiche Classen. Durchläuft A die h_α verschiedenen Elemente der α^{ten} Classe, B die h_β Elemente der β^{ten} und C die h_γ Elemente der γ^{ten}, so soll die Zahl $h_{\alpha\beta\gamma}$ (die auch Null sein kann) angeben, wie viele der $h_\alpha h_\beta h_\gamma$ Elemente ABC gleich dem Hauptelemente sind, also der Gleichung

$$ABC = E \tag{3.}$$

genügen. Da dann $AB = C^{-1}$ ist, so giebt $h_{\alpha\beta\gamma}$ auch an, wie viele der $h_\alpha h_\beta$ Elemente AB der Classe (γ) angehören. Die Gleichung (3.) ist identisch mit $BCA = E$ und $CAB = E$. Daher sind auch $h_{\alpha\beta\gamma}$ der $h_\beta h_\gamma$ Producte BC in (α') und $h_{\alpha\beta\gamma}$ der $h_\gamma h_\alpha$ Producte CA in (β') enthalten. Mithin ist $h_{\alpha\beta\gamma}$ nicht grösser als die kleinste der drei Zahlen $h_\beta h_\gamma$, $h_\gamma h_\alpha$ und $h_\alpha h_\beta$.

An old problem

Suppose $G = \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C}), G_2(\mathbb{C}), F_4(\mathbb{C}), \ldots$

This is called the ‘Deligne–Simpson problem’





Products of Matrices

CARLOS T. SIMPSON

Introduction

We will consider the matrix equation

$$A_1 \cdots A_k = 1$$

for matrices A_i in specified conjugacy classes $C_i \subset SL(n, \mathbb{C})$. The question is whether there exists a solution.

THEOREM. *Suppose C_k is semisimple with generic distinct eigenvalues. Then there exists a solution $A_1 \cdots A_k = 1$ if and only if $\sum \dim(C_i) \geq 2n^2 - 2$ and $r_1 + \cdots + r_{k-1} \geq n$.*

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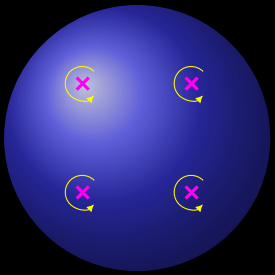
COUNTING POINTS ON GENERIC CHARACTER VARIETIES

STEFANO GIANNINI[1], MASOUD KAMGARPOUR[2], GYEONGHYEON NAM[3], AND BAILEY WHITBREAD[4]

Deligne–Simpson problems

$X = \mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \dots, p_r\}$ $\qquad$ $\pi_1(X) = \langle \ell_1, \dots, \ell_r \ : \ \ell_1 \cdots \ell_r = 1 \rangle$



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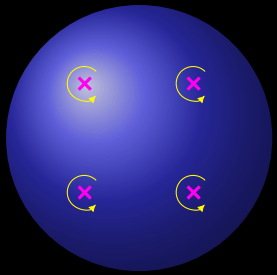

Tuples $(A_1, \dots, A_r)$ in $\mathrm{GL}_n(\mathbb{C})$
with $A_1 \cdots A_r = 1$ and $A_i \in C_i$



Local systems on X
with monodromy C_i
around p_i

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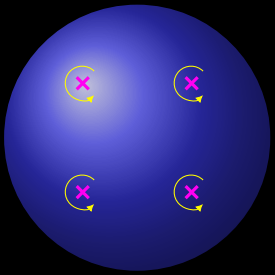

Tuples $(A_1, \dots, A_r)$ in $\mathrm{GL}_n(\mathbb{C})$
with $A_1 \cdots A_r = 1$ and $A_i \in C_i$

$\updownarrow$

GL_n-connections on X
with reg. sing. at p_i
and monodromy in C_i

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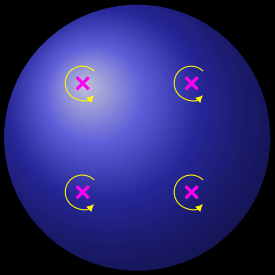

$\nabla = d + B(t)\,dt$

$B(t) \in \mathfrak{gl}_n(\mathbb{C}((t)))$

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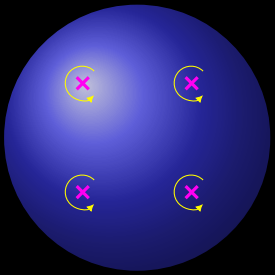

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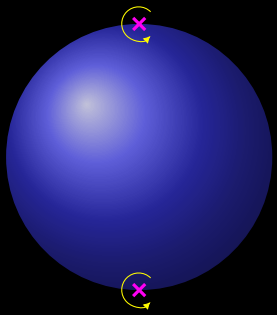
Tuples $(A_1, \dots, A_r)$ in $G(\mathbb{C})$
with $A_1 \cdots A_r = 1$ and $A_i \in C_i$

$\Updownarrow$

G-connections on X
with reg. sing. at p_i
and monodromy in C_i

Deligne–Simpson problems

$X = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$



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with reg. sing. at ∞
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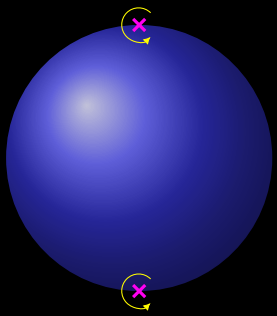

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$\mathbb{C}$-points on
$\mathcal{M}(C, \beta)$

$\updownarrow$

G-connections on X
with reg. sing. at ∞
and irreg. sing. at 0

The braid variety

Conj. class C in $G(\mathbb{C})$ Pos. braid $\beta = \widetilde{w_1} \cdots \widetilde{w}_r$ in $B(G)$

The braid variety is

$$\mathcal{M}(C, \beta) = \left\{ (g, F_1, \ldots, F_r) \ : \ \begin{array}{c} g \in C, \ F_i \in G/B \\ F_i \xrightarrow{w_i} F_{i+1}, \ F_r \xrightarrow{w_r} gF_1 \end{array} \right\}$$

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G-connections on X with reg. sing. at ∞ and irreg. sing. at 0 $\quad \rightsquigarrow \quad$ $\mathbb{C}$-points on $\mathcal{M}(C, \beta)$

The braid variety

Conj. class $[\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)]$ in $\mathrm{GL}_2(\mathbb{C})$ Pos. braid $\beta = \widetilde{(1\,2)}\widetilde{(1\,2)}$ in B_2

The braid variety is

$$\mathcal{M}(C,\beta) = \left\{ (g, F_1, F_2) \ : \begin{array}{c} g \in [\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)],\ F_i \in G/B \\ F_1 \xrightarrow{(1\,2)} F_2 \xrightarrow{(1\,2)} gF_1 \end{array} \right\}$$

GL_2-connections on X with reg. sing. at ∞ and irreg. sing. at 0 $\leftrightsquigarrow$ $\mathbb{C}$-points on $\mathcal{M}(C,\beta)$

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The braid variety is

$$\mathcal{M}(C,\beta) = \left\{ (g,\ell_1,\ell_2) \ : \ \begin{array}{c} g \in [\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)],\ \ell_i \in \mathbb{P}^1 \\ \ell_1 \neq \ell_2 \neq g\ell_1 \end{array} \right\}$$

GL_2-connections on X with reg. sing. at ∞ and irreg. sing. at 0 $\longleftrightarrow$ $\mathbb{C}$-points on $\mathcal{M}(C,\beta)$

The braid variety

Conj. class $[\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)]$ in $\mathrm{GL}_2(\mathbb{F}_q)$ Pos. braid $\beta = \widetilde{(1\,2)}\widetilde{(1\,2)}$ in B_2

The braid variety is

$$\mathcal{M}(C,\beta)(\mathbb{F}_q) = \left\{(g,\ell_1,\ell_2) \ : \ \begin{array}{c} g \in [\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)],\ \ell_i \in \mathbb{P}^1(\mathbb{F}_q) \\ \ell_1 \neq \ell_2 \neq g\ell_1 \end{array}\right\}$$

$|\mathcal{M}(C,\beta)(\mathbb{F}_q)|$ is a non-zero poly. in q $\rightsquigarrow$ $\mathbb{C}$-points on $\mathcal{M}(C,\beta)$ exists

The shape of our theorem

Simple gp. $G/\mathbb{C}$ Conj. class C in G Pos. braid β	$\rightsquigarrow$	Is there a G-conn. with local behavior C and β at ∞ and 0?
(G, C, β)	$\mapsto$	yes/no

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Simple gp. $G/\mathbb{C}$ Conj. class C in G Pos. braid β	$\rightsquigarrow$	Is there a G-conn. with local behavior C and β at ∞ and 0?
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Theorem (Kamgarpour–W., ‘26)
If G is exceptional, check our table

Theorem (Jakob–Yun, ‘23)
If G is classical, check our table

The shape of our theorem

Simple gp. $G/\mathbb{C}$ Conj. class C in G Pos. braid β	$\rightsquigarrow$	Is there a $\mathbb{C}$-point on $\mathcal{M}(C,\beta)$?
(G, C, β)	$\mapsto$	yes/no

Theorem (Kamgarpour–W., '26)
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The shape of our theorem

Simple gp. $G/\mathbb{C}$ Conj. class C in G Pos. braid β	$\rightsquigarrow$	Is $\lvert\mathcal{M}(C,\beta)(\mathbb{F}_q)\rvert$ a non-zero polynomial in q?
(G, C, β)	$\mapsto$	yes/no

Theorem (Kamgarpour–W., '26)
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Theorem (Jakob–Yun, '23)
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The shape of our proof

Fact: $\mathcal{M}(C, \beta) \simeq C \cap BwB$ when $\beta = \widetilde{w}$

In 2011, Lusztig gave a formula for counting points over $\mathbb{F}_q$:

$$|(C \cap BwB)(\mathbb{F}_q)| = \frac{1}{|G(\mathbb{F}_q)|} \sum_{g \in C(\mathbb{F}_q)} \sum_{E \in \mathrm{Irr}(W)} \mathrm{tr}(g, U_E) \, \mathrm{tr}(T_w, H_E)$$

where

U_E is a unipot. $G(\mathbb{F}_q)$-rep. & H_E is an irred. Hecke alg. rep.