A positivity conjecture for character varieties

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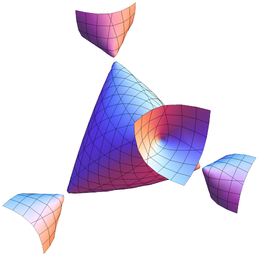

with Stefano Giannini, Masoud Kamgarpour & GyeongHyeon Nam

Character varieties

This talk is about two varieties:

$$X := \text{multiplicative character variety}$$

$$Y := \text{additive character variety}$$

X is built from reductive groups $G = \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, etc.

Y is built from their Lie algebras $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, etc.

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Main Theorem

$|\mathsf{X}(\mathbb{F}_q)|$ and $|\mathsf{Y}(\mathbb{F}_q)|$ are polynomials in q

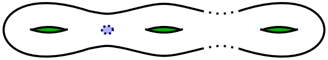
Character varieties

$G :=$ reductive group over $\mathbb{F}_q$

$\mathfrak{g} :=$ its Lie algebra over $\mathbb{F}_q$

$\mathcal{C} :=$ conjugacy class in G

$\mathcal{O} :=$ adjoint orbit in $\mathfrak{g}$

$\pi_1\Big($  $\Big) = \left\langle a_1, b_1, \dots, a_g, b_g, c \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\rangle$

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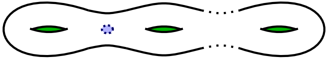
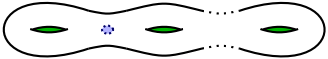
$\pi_1\Big($  $\Big) = \left\langle a_1, b_1, \ldots, a_g, b_g, c \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\rangle$

$\Big\{ f\colon \pi_1\Big($  $\Big) \to G \;\Big|\; f(c) \in \mathcal{C} \Big\} \Big/ G$

$$\Big\updownarrow$$

$$\left\{ (A_1, B_1, \ldots, A_g, B_g, C) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i]\, C = 1 \right\} \Big/ G$$

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The additive character variety is

$$\mathsf{Y} := \left\{ (X_1, Y_1, \ldots, X_g, Y_g, Z) \in \mathfrak{g}^{2g} \times \mathcal{O} \;\middle|\; \sum_{i=1}^{g} [X_i, Y_i] + Z = 0 \right\} \Big/ G$$

The GL_2 example

$G :=$ general linear group GL_2

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If $g = 1$, the multiplicative and additive character varieties are

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One can compute by hand

$$|\mathsf{X}(\mathbb{F}_q)| = q^4 - q^3 - q + 1 \quad \text{and} \quad |\mathsf{Y}(\mathbb{F}_q)| = q^4 + q^3$$

Our positivity conjecture

New examples of $|Y(\mathbb{F}_q)|$:

$q^2 + 6q$

$q^4 + 6q^3 + 20q^2$

$q^8 + 6q^7 + 19q^6 + 45q^5 + 99q^4$

$q^6 + 2q^5 + 2q^4 + q^3$

$q^8 + 2q^7 + 4q^6 + 4q^5 + q^4$

$q^{12} + 2q^{11} + 3q^{10} + 5q^9 + \cdots$

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Main Conjecture

(i) $|Y(\mathbb{F}_q)|$ *has positive coefficients*

(ii) $H^*(Y)$ *is the 'pure' subring of* $H^*(X)$

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Counting points

We access cohomology by counting points over finite fields

Weil conjectures

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GL_2-example: $|X(\mathbb{F}_q)| = q^4 - q^3 - q + 1$ & $|Y(\mathbb{F}_q)| = q^4 + q^3$

$$\Downarrow$$

$\dim(X) = \dim(Y) = 4$, $\chi(X) = 0$, $\chi(Y) = 2$, X & Y connected

Patterns and observations

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Corollary

$|Y(\mathbb{F}_q)|$ has positive coefficients in 160,000+ cases:

(i) *When* $\mathrm{rank}(G) \leq 6$,

(ii) *When*  *has at most genus 10, and*

(iii) *When*  *has at most 1000 punctures* julia

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