A purity conjecture for character varieties

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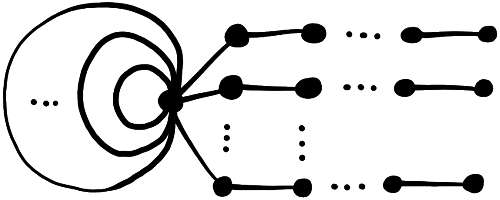


with S. Giannini, M. Kamgarpour, G. Nam

Character varieties

This talk is about two varieties:

$$\mathsf{X} := \text{multiplicative character variety}$$

$$\mathsf{Y} := \text{additive character variety}$$

X is built from reductive groups $G = \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, etc.

Y is built from their Lie algebras $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, etc.

Main Theorem (GKNW '25)
$|\mathsf{X}(\mathbb{F}_q)|$ and $|\mathsf{Y}(\mathbb{F}_q)|$ are polynomials in q

Character varieties

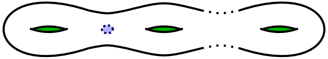
$G :=$ reductive group over $\mathbb{F}_q$

$\mathfrak{g} :=$ its Lie algebra over $\mathbb{F}_q$

$\mathcal{C} :=$ conjugacy class in G

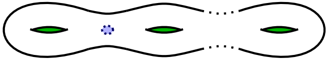
$\mathcal{O} :=$ adjoint orbit in $\mathfrak{g}$

$$\pi_1\Big($$



$$\Big) = \left\langle a_1, b_1, \ldots, a_g, b_g, c \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\rangle$$

$$\Big\{ f \colon \pi_1\Big($$



$$\Big) \to G \;\Big|\; f(c) \in \mathcal{C} \Big\} \Big/ G$$

$$\updownarrow$$

$$\left\{ (a_1, b_1, \ldots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\} \Big/ G$$

Character varieties

The multiplicative character variety is

$$\mathsf{X} := \left\{ (a_1, b_1, \ldots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\} \Big/ G$$

The additive character variety is

$$\mathsf{Y} := \left\{ (x_1, y_1, \ldots, x_g, y_g, z) \in \mathfrak{g}^{2g} \times \mathcal{O} \;\middle|\; \sum_{i=1}^{g} [x_i, y_i] + z = 0 \right\} \Big/ G$$

Our purity conjecture

Examples of $|\mathsf{Y}(\mathbb{F}_q)|$:

$q^2 + 6q$

$q^4 + 6q^3 + 20q^2$

$q^8 + 6q^7 + 19q^6 + 45q^5 + 99q^4$

$q^6 + 2q^5 + 2q^4 + q^3$

$q^8 + 2q^7 + 4q^6 + 4q^5 + q^4$

$q^{12} + 2q^{11} + 3q^{10} + 5q^9 + \cdots$

Main Conjecture

(a) $|\mathsf{Y}(\mathbb{F}_q)| \in \mathbb{N}[q]$

(b) *The cohomology of* Y *is pure*

(c) *The cohomology of* Y *sits inside the cohomology of* X

Purity

$$H^*_{\mathrm{pure}}(Y) \hookrightarrow H^*(Y) \qquad H^*_{\mathrm{pure}}(X) \hookrightarrow H^*(X)$$

Theorem (Hausel–Letellier–Rodriguez-Villegas)

If $G = \mathrm{GL}_n$ then the cohomology of Y is pure

Conjecture (Hausel–Letellier–Rodriguez-Villegas)

If $G = \mathrm{GL}_n$ then $H^*(Y) \simeq H^*_{pure}(X)$

$$H^*_{\mathrm{pure}}(Y) \xhookrightarrow{\sim} H^*(Y) \overset{\sim}{\dashrightarrow} H^*_{\mathrm{pure}}(X) \hookrightarrow H^*(X)$$

Kac polynomials

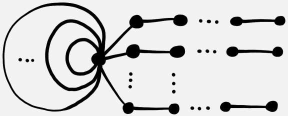
When $G = \mathrm{GL}_n$, the polynomials $|\mathsf{Y}(\mathbb{F}_q)|$ are Kac polynomials

$$\begin{array}{c}\text{quiver } Q \\ \text{vector } \mathbf{v}\end{array} \quad \rightsquigarrow \quad K(q) := \#\left\{\begin{array}{c}\text{abs. indec. dim } \mathbf{v} \\ Q\text{-reps over } \mathbb{F}_q\end{array}\right\} \Big/ \text{iso.}$$

Theorem (Kac)
$K(q) \in \mathbb{Z}[q]$

Theorem (HLRV)
$K(q) \in \mathbb{N}[q]$

Theorem (Crawley-Boevey)
If $G = \mathrm{GL}_n$ then Y is a quiver variety and $|\mathsf{Y}(\mathbb{F}_q)| = K(q)$



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