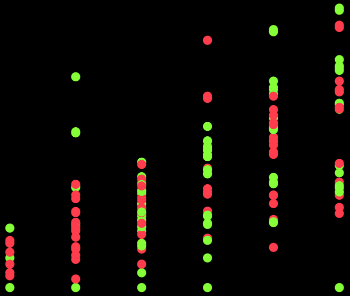
Spectroscopy for knots

Bailey Whitbread
University of Sydney

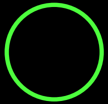


Detecting the unknot

Open Problem

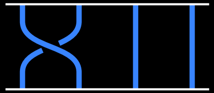
Does the Jones polynomial detect the unknot?
i.e., does $J(K) = J(\textit{unknot})$ *imply* $K \simeq \textit{unknot}$*?*

$$J\colon \{\text{Knots \& links}\}_{/\simeq} \to \mathbb{Z}[q, q^{-1}]$$

$J\Big($  $\Big) = 1$ $J\Big($  $\Big) = q^{-1} + q^{-3} - q^{-4}$

Knots and braids

Braids are the algebraic analogues of knots

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \ : \ \text{‘braid relations’} \right\rangle \quad \sigma_1 = $$  $$\in B_4$$

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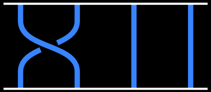
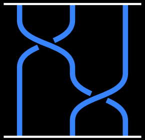
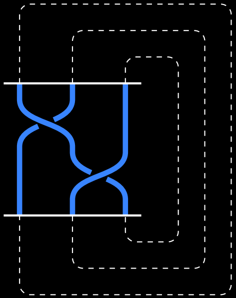
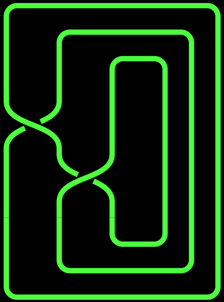
$\sigma_1 =$  $\in B_4$

A bridge between braids and knots is the braid closure:

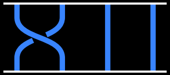
$\beta =$  $\rightsquigarrow$  $\rightsquigarrow$  $= \widehat{\beta}$

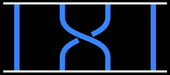
The Burau representation

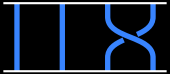
Burau rep.

$$B_4 \to \mathrm{GL}_3(\mathbb{Z}[v^{\pm 1}])$$

$$B_4 = \left\langle \sigma_1, \sigma_2, \sigma_3 \;:\; \text{‘braid relations’} \right\rangle$$

$\sigma_1 =$  $\mapsto \begin{pmatrix} -v^2 & -v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\sigma_2 =$  $\mapsto \begin{pmatrix} 1 & 0 & 0 \\ -v & -v^2 & -v \\ 0 & 0 & 1 \end{pmatrix}$

$\sigma_3 =$  $\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v & -v^2 \end{pmatrix}$

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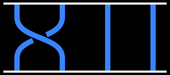
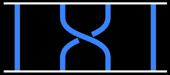
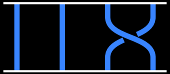
$\sigma_1 =$  $\mapsto \begin{pmatrix} -v^2 & -v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\sigma_2 =$  $\mapsto \begin{pmatrix} 1 & 0 & 0 \\ -v & -v^2 & -v \\ 0 & 0 & 1 \end{pmatrix}$

$\sigma_3 =$  $\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v & -v^2 \end{pmatrix}$

Theorem (Birman, Bigelow, Ito)

Jones poly. does not detect unknot $\leftsquigarrow$ Burau rep. of B_4 is unfaithful $\leftrightsquigarrow$ The matrices

$$\begin{pmatrix} -v^{-2} & -v^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & -v & -v^2 \end{pmatrix} \;\&\; \begin{pmatrix} 0 & v^{-3} & v^{-2} \\ 0 & -v^{-2} & v^4 - v^{-1} \\ 1 & v^{-1} & 1 - v^2 \end{pmatrix}$$

satisfy a non-trivial relation

Detecting relations

Goal: Determine if $\begin{pmatrix} -v^{-2} & -v^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & -v & -v^2 \end{pmatrix}$ & $\begin{pmatrix} 0 & v^{-3} & v^{-2} \\ 0 & -v^{-2} & v^4-v^{-1} \\ 1 & v^{-1} & 1-v^2 \end{pmatrix}$ have a relation

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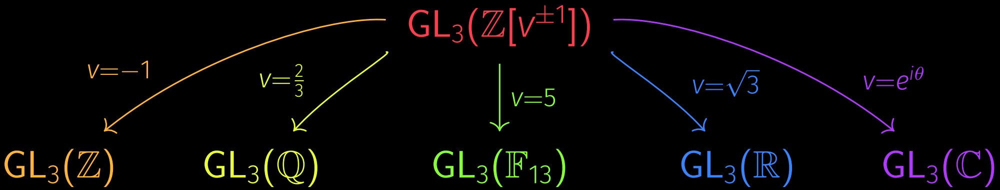
Key idea: If a relation exists, it must exist in all specialisations



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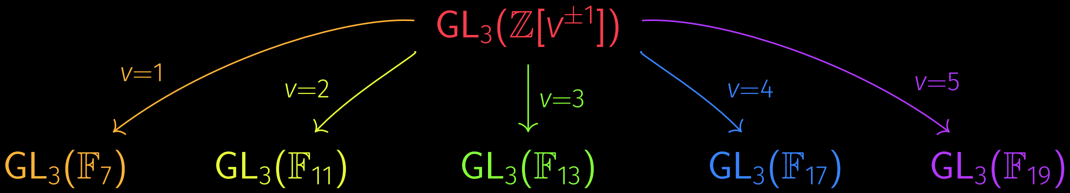

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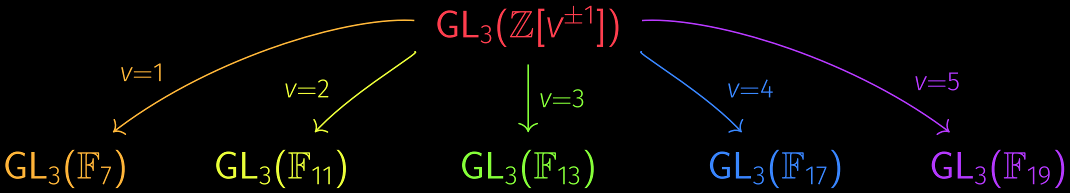

Subgoal: Detect relations between A & B in $GL_3(\mathbb{F}_{13})$

An experiment in $\mathrm{SL}_2(\mathbb{F}_{13})$

Experiment: Detect relations between A & B in $\mathrm{SL}_2(\mathbb{F}_{13})$

Method: Use the natural action $\mathrm{SL}_2(\mathbb{F}_{13}) \curvearrowright \mathbb{P}^1(\mathbb{F}_{13})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad p = [x : y] \qquad g \cdot p = [ax + by : cx + dy]$$

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Perm. representation: $\quad g \mapsto P_g \quad$ permutation matrix

$$\mathrm{tr}(P_g) = \begin{cases} 14 & \text{if } g = \pm\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \\ 0,\ 1,\ 2 & \text{otherwise} \end{cases} \quad \rightsquigarrow$$

$\mathrm{tr}(P_g)$ detects if g is the identity

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$$\mathrm{tr}({P_g}^{\otimes 4}) = \begin{cases} 14^4 & \text{if } g = \pm \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ 0^4,\ 1^4,\ 2^4 & \text{otherwise} \end{cases} \quad \rightsquigarrow$$

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$$(P_A + P_{A^{-1}})(P_B + P_{B^{-1}}) = P_{AB} + P_{AB^{-1}} + P_{A^{-1}B} + P_{A^{-1}B^{-1}}$$

$$((P_A + P_{A^{-1}})(P_B + P_{B^{-1}}))^2 = P_{ABAB} + P_{ABAB^{-1}} + P_{ABA^{-1}B} + P_{ABA^{-1}B^{-1}} + \cdots$$

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$$((P_A + P_{A^{-1}})(P_B + P_{B^{-1}}))^\ell = \sum_{\substack{\text{words } W \\ \text{length}=2\ell}} P_W$$

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Experiment: Detect relations between A & B in $\mathrm{SL}_2(\mathbb{F}_p)$

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$$\mathrm{tr}(((P_A + P_{A^{-1}})(P_B + P_{B^{-1}}))^\ell) = \sum_{\substack{\text{words } W \\ \text{length}=2\ell}} \mathrm{tr}(P_W)$$

An experiment in $SL_2(\mathbb{F}_{13})$

$A = \begin{pmatrix} 2 & 0 \\ 12 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & 3 \\ 10 & 3 \end{pmatrix}$

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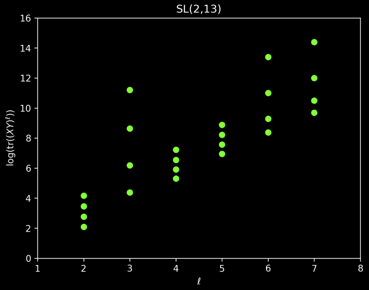

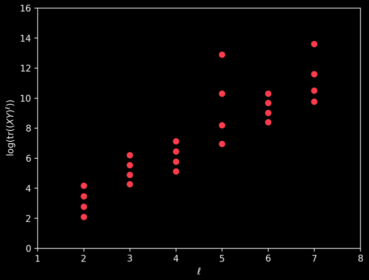

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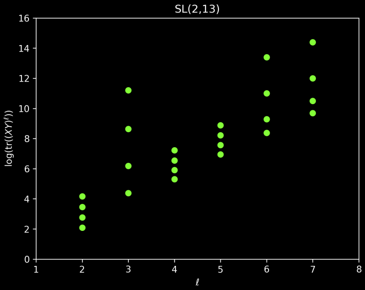

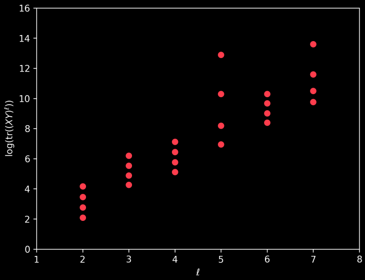

Shortest relation

$AB^{-1}AB^{-1}AB^{-1}$

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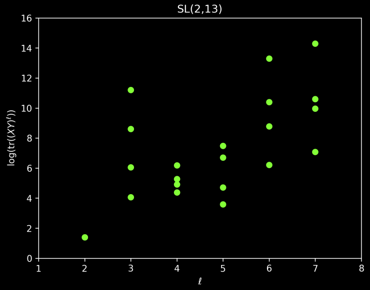

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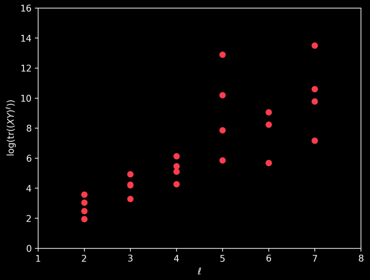

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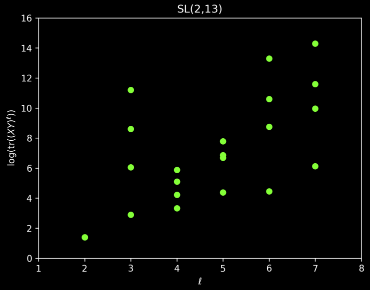

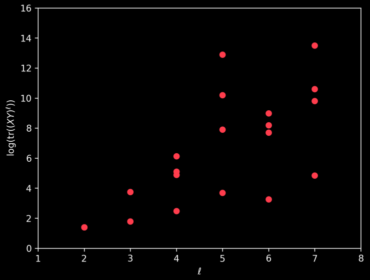

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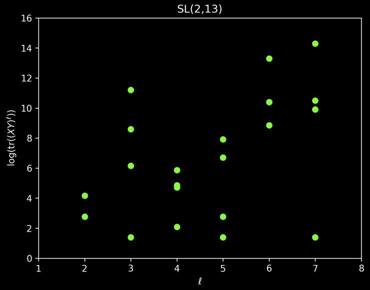

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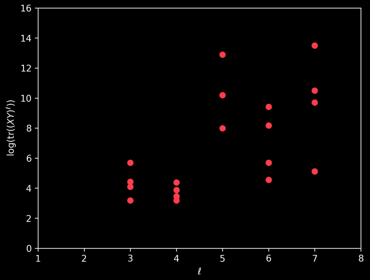

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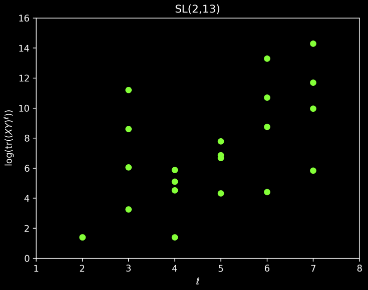

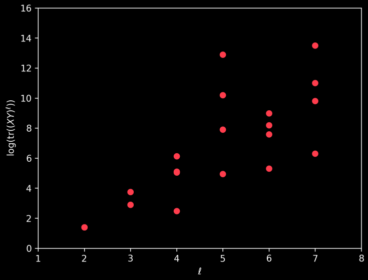

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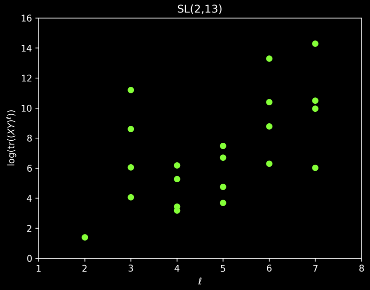

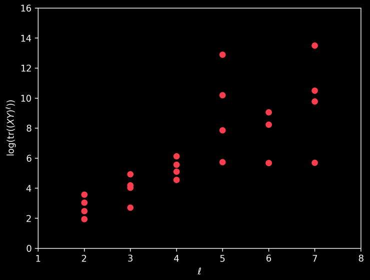

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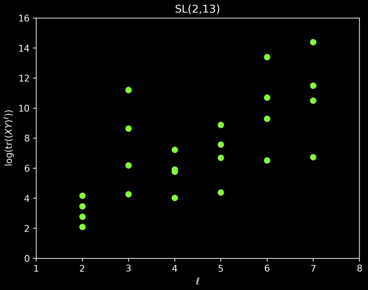

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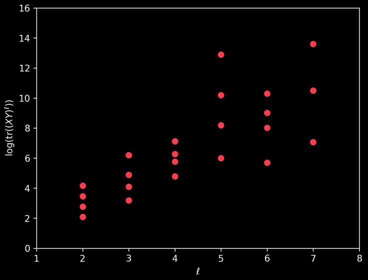

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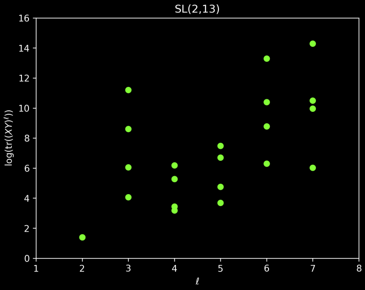

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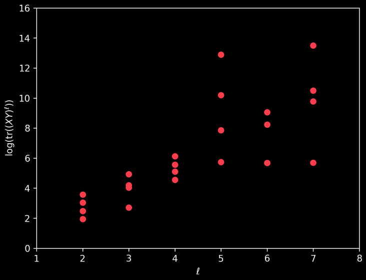

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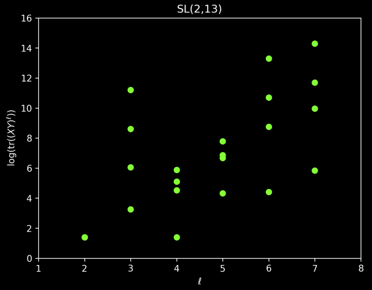

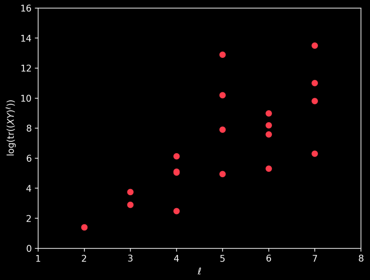

Shortest relation

$AB^{-1}AB^{-1}AB^{-1}$

Shortest relation

$A^{-1}B^{-1}A^{-1}BAB^{-1}A^{-1}B^{-1}AB$

An experiment in $SL_2(\mathbb{F}_{13})$

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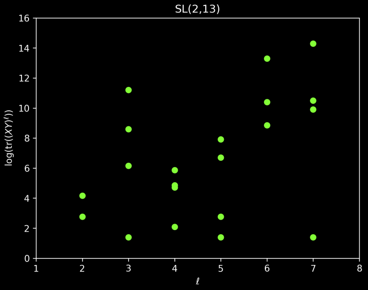

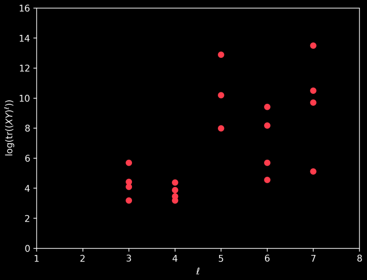

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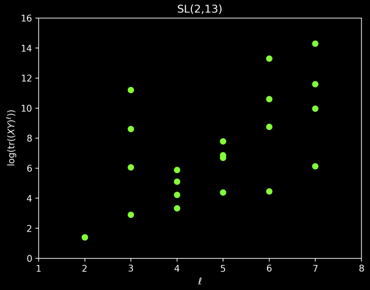

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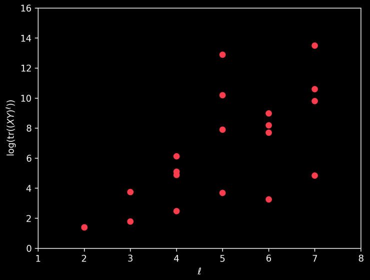

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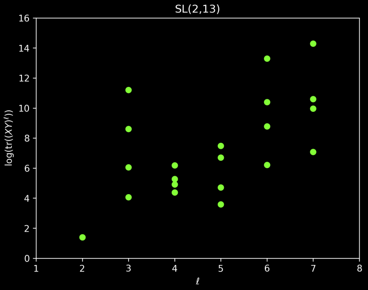

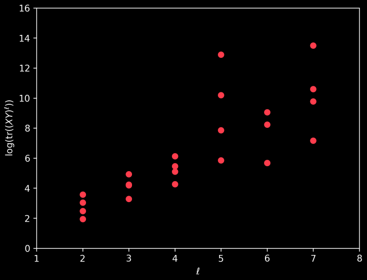

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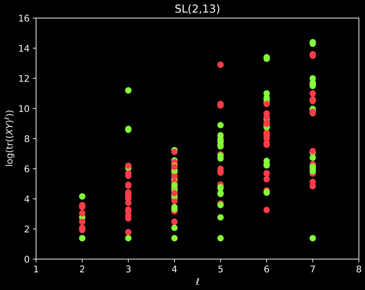

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