Counting points on character varieties

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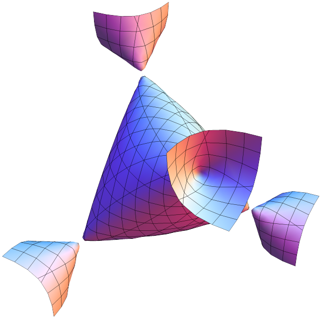


Character varieties

This talk is about two varieties:

$$\mathsf{X} := \text{multiplicative character variety}$$

$$\mathsf{Y} := \text{additive character variety}$$

X is built from groups like $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, etc.

Y is built from Lie algebras like $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, etc.

Main Theorem

$|\mathsf{X}(\mathbb{F}_q)|$ *and* $|\mathsf{Y}(\mathbb{F}_q)|$ *are polynomials in* q

Character varieties

Fix: A 'big' conjugacy class $\mathcal{C}$ in G

The multiplicative character variety is

$$X := \left\{(a_1, b_1, \dots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \,\middle|\, \prod_{i=1}^{g} [a_i, b_i]\, c = 1\right\} \Big/ G$$

Fix: A 'big' orbit $\mathcal{O}$ in $\mathfrak{g}$

The additive character variety is

$$Y := \left\{(x_1, y_1, \dots, x_g, y_g, z) \in \mathfrak{g}^{2g} \times \mathcal{O} \,\middle|\, \sum_{i=1}^{g} [x_i, y_i] + z = 0\right\} \Big/ G$$

A GL_2 example

If $G = \mathrm{GL}_2$ and $g = 1$ then

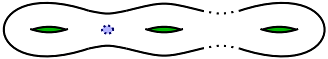
$$\mathsf{X} = \left\{ (a, b, c) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathcal{C} \;\middle|\; [a, b]\, c = 1 \right\} \Big/ \mathrm{GL}_2$$

$$\mathsf{Y} = \left\{ (x, y, z) \in \mathfrak{gl}_2 \times \mathfrak{gl}_2 \times \mathcal{O} \;\middle|\; [x, y] + z = 0 \right\} \Big/ \mathrm{GL}_2$$

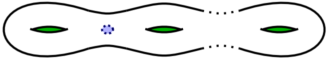
One computes by hand

$$|\mathsf{X}(\mathbb{F}_q)| = q^4 - q^3 - q + 1, \quad |\mathsf{Y}(\mathbb{F}_q)| = q^4 + q^3$$

Character varieties in nature

$\pi_1\Big($  $\Big) = \left\langle a_1, b_1, \ldots, a_g, b_g, c \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\rangle$

Fact: There is a bijection

$\Big\{ f\colon \pi_1\Big($  $\Big) \to G \;\Big|\; f(c) \in \mathcal{C} \Big\} \Big/ G$

$$\updownarrow$$

$$\left\{ (a_1, b_1, \ldots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\} \Big/ G$$

Purity

$$H^*_{\mathrm{pure}}(Y) \hookrightarrow H^*(Y) \qquad H^*_{\mathrm{pure}}(X) \hookrightarrow H^*(X)$$

Theorem (Hausel–Letellier–Rodriguez-Villegas)

If $G = \mathrm{GL}_n$ then $H^*_{pure}(Y) = H^*(Y)$

Conjecture (Hausel–Letellier–Rodriguez-Villegas)

If $G = \mathrm{GL}_n$ then $H^*(Y) \simeq H^*_{pure}(X)$

$$H^*_{\mathrm{pure}}(Y) \xrightarrow{\sim} H^*(Y) \overset{\sim}{\dashrightarrow} H^*_{\mathrm{pure}}(X) \hookrightarrow H^*(X)$$

Counting points

We access cohomology by counting points over finite fields

Weil conjectures

$$|X(\mathbb{F}_q)| \rightsquigarrow H^*(X)$$

$$|Y(\mathbb{F}_q)| \rightsquigarrow H^*(Y)$$

GL_2-example: $|X(\mathbb{F}_q)| = q^4 - q^3 - q + 1, \quad |Y(\mathbb{F}_q)| = q^4 + q^3$

Patterns and observations

Main Theorem

$|X(\mathbb{F}_q)|$ *and* $|Y(\mathbb{F}_q)|$ *are polynomials in q with explicit formulas*

Theorem

$|X(\mathbb{F}_q)|$ *is always palindromic*

E.g. $|X(\mathbb{F}_q)| = q^4 - q^3 - q + 1$

$\rightsquigarrow 1, -1, 0, -1, 1$

Theorem

We know the dimensions, # of components and Euler characteristics of X and Y

Theorem

$|Y(\mathbb{F}_q)| \in \mathbb{N}[q]$ *when* rank $G \leq 4$ *and* $g \leq 10$

Conjecture

$|Y(\mathbb{F}_q)| \in \mathbb{N}[q]$ *always*

Conjecture

$H^*_{pure}(Y) = H^*(Y)$

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