Character varieties

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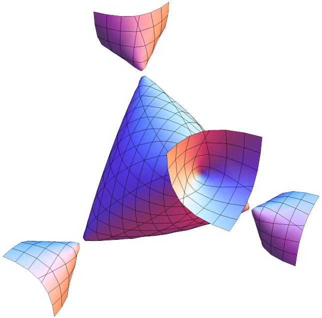


An old problem

(G finite gp.) + (conj. classes C_1, C_2, C_3)

Question: What is the # of triples

$$(x, y, z) \in C_1 \times C_2 \times C_3$$

satisfying $xyz = 1$.

Answer: Rep. theory of G.

$$G \rightsquigarrow \mathrm{Rep}(G) = \{\chi \colon G \to \mathbb{C}\}$$

$$\sum_{\chi \in \mathrm{Rep}(G)} \frac{\chi(C_1)\,\chi(C_2)\,\chi(C_3)}{\chi(1)^3}$$

An old problem

53.

Über Gruppencharaktere

Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin
985–1021 (1896)

p, ℓ prime

$px + \ell \quad x = 1, 2, 3, \ldots$

Bei dem Beweise des Satzes, dass jede lineare Function einer Variabeln unendlich viele Primzahlen darstellt, wenn ihre Coefficienten theilerfremde ganze Zahlen sind, benutzte Dirichlet zum ersten Male gewisse Systeme von Einheitswurzeln, die auch in der nahe verwandten Frage nach der Anzahl der Idealclassen in einem Kreiskörper auftreten (vergl. die Bemerkung von Dedekind in Dirichlet's Vorlesungen über Zahlentheorie, 4. Aufl. S. 625), sowie bei der Verallgemeinerung jenes Satzes auf quadratische Formen und in den Untersuchungen über deren Eintheilung in Geschlechter. Die charakteristische Eigenschaft dieser Ausdrücke besteht nach Dedekind darin, dass sie von einer variabeln positiven ganzen Zahl n abhängige Grössen $\chi(n)$ sind, die nur eine endliche Anzahl von Werthen haben und der Bedingung

$$\chi(m)\chi(n) = \chi(mn)$$

genügen. Wie er in rein abstracter Form ausführt, lassen sich den Elementen $A, B, C, \cdots$ jeder endlichen Gruppe $\mathfrak{H}$ vertauschbarer Elemente (Abel'schen Gruppe) solche Einheitswurzeln $\chi(A), \chi(B), \chi(C), \cdots$ zuordnen, welche die Gleichungen

$$\chi(A)\chi(B) = \chi(AB)$$

befriedigen, und die er nach dem Vorgange von Gauss *die Charaktere der Gruppe* nannte.

Seien (α), (β), (γ) irgend drei verschiedene oder gleiche Classen. Durchläuft A die h_α verschiedenen Elemente der α^{ten} Classe, B die h_β Elemente der β^{ten} und C die h_γ Elemente der γ^{ten}, so soll die Zahl $h_{\alpha\beta\gamma}$ (die auch Null sein kann) angeben, wie viele der $h_\alpha h_\beta h_\gamma$ Elemente ABC gleich dem Hauptelemente sind, also der Gleichung

$$(3.) \qquad ABC = E$$

genügen. Da dann $AB = C^{-1}$ ist, so giebt $h_{\alpha\beta\gamma}$ auch an, wie viele der $h_\alpha h_\beta$ Elemente AB der Classe (γ') angehören. Die Gleichung (3.) ist identisch mit $BCA = E$ und $CAB = E$. Daher sind auch $h_{\alpha\beta\gamma}$ der $h_\beta h_\gamma$ Producte BC in (α') und $h_{\alpha\beta\gamma}$ der $h_\gamma h_\alpha$ Producte CA in (β') enthalten. Mithin ist $h_{\alpha\beta\gamma}$ nicht grösser als die kleinste der drei Zahlen $h_\beta h_\gamma$, $h_\gamma h_\alpha$ und $h_\alpha h_\beta$.

Building blocks: algebraic groups

Fix: Field $k = \mathbb{R},\ \mathbb{C}, \ldots$

Def: (Alg. gp.) $=$ (gp.) $+$ (subspace of k^n cut out by poly's).[1]

(i) **Poly:** $0 \in k[x]$
Alg. gp.: $(k, +)$

(ii) **Poly:** $xy - 1 \in k[x, y]$
Alg. gp.: $(k^\times, \times)$

(iii) **Poly:** $ad - bc - 1$
Alg. gp.: $\mathrm{SL}_2(k)$

(iv) **Poly:** $t(ad - bc) - 1$
Alg. gp.: $\mathrm{GL}_2(k)$

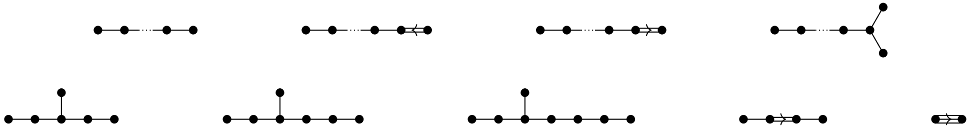
[1] In this talk, alg. gp $\equiv$ *affine* alg. gp.

Building blocks: algebraic groups

$$(v)\;\; Q = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \rightsquigarrow \left\{ A \in \mathrm{GL}_4(k) \;:\; AQA^{tr} = Q \right\} =: \mathrm{Sp}_4(k).$$

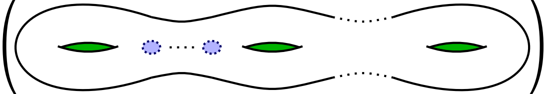
Theorem: All alg. gps are subgroups of GL_n.

$$\mathrm{GL}_n, \quad \mathrm{SL}_n, \quad \mathrm{Sp}_{2n}, \quad \mathrm{SO}_n, \quad \ldots$$



The spaces I study: character varieties

Fix: Surface S with genus $g \geq 0$ and $r \geq 1$ punctures.

$\pi_1(S) = \pi_1\Big($  $\Big)$

$$= \frac{\langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_r \rangle}{[a_1, b_1] \cdots [a_g, b_g]\, y_1 \cdots y_r}$$

Fix: (Alg. gp. G) + (Conj. classes $C_1, \ldots, C_r \subseteq G$).

$$\mathbf{R} := \left\{ f\colon \pi_1(S) \to G \;\middle|\; f(y_i) \in C_i \right\} \curvearrowleft G \text{ by conj.}$$

$\rightsquigarrow$ character variety $\mathbf{R}/G$

The spaces I study: character varieties

$$f \longleftrightarrow (\ f(a_1),\ f(b_1), \ldots, f(a_g),\ f(b_g),\ f(y_1), \ldots, f(y_r)\ )$$

Example: Surface S with genus $g = 1$ and $r = 3$ punctures.

$$\pi_1(S) = \left\langle a, b, y_1, y_2, y_3 \,\middle|\, [a, b]\, y_1\, y_2\, y_3 = 1 \right\rangle$$

$$\mathrm{R} = \left\{ \begin{array}{c} (A, B, Y_1, Y_2, Y_3) \in G^5 \text{ such that} \\ [A, B]\, Y_1\, Y_2\, Y_3 = 1 \text{ and } Y_i \in C_i \end{array} \right\}$$

How do I study a space?

NUMBERS OF SOLUTIONS OF EQUATIONS IN FINITE FIELDS

ANDRÉ WEIL

The equations to be considered here are those of the type

$$a_0x_0^{n_0} + a_1x_1^{n_1} + \cdots + a_rx_r^{n_r} = b. \tag{1}$$

Such equations have an interesting history. In art. 358 of the *Disquisitiones* [1 a],[1] Gauss determines the Gaussian sums (the so-called cyclotomic "periods") of order 3, for a prime of the form $p=3n+1$, and at the same time obtains the numbers of solutions for all congruences $ax^3-by^3\equiv 1 \pmod p$. He draws attention himself to the elegance of his method, as well as to its wide scope; it is only much later, however, viz. in his first memoir on biquadratic residues [1b], that he gave in print another application of the same method; there he treats the next higher case, finds the number of solutions of any congruence $ax^4-by^4\equiv 1 \pmod p$, for a prime of the form $p=4n+1$, and derives from this the biquadratic character of 2 mod p, this being the ostensible purpose of the whole highly ingenious and intricate investigation. As an incidental consequence ("*coronidis loco*," p. 89), he also gives in substance the number of solutions of any congruence $y^2\equiv ax^4-b \pmod p$; this result includes as a special case the theorem stated as a conjecture ("*observatio per inductionem facta gravissima*") in the last entry of his *Tagebuch* [1c];[2] and it implies the truth of what has lately become known as the Riemann hypothesis, for the function-field defined by that equation over the prime field of p elements.

Fix: A space $\mathbf{X}$ defined by poly's with $\mathbb{Z}$-coefficients

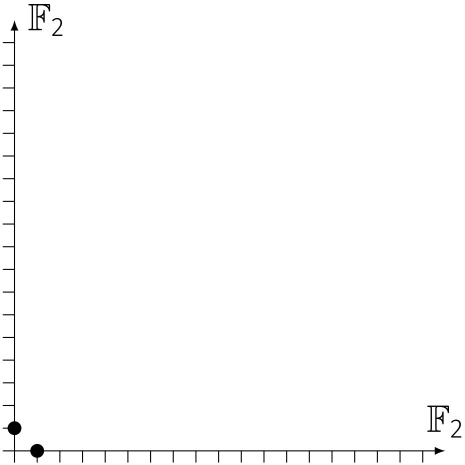
Moral: Count $|\mathbf{X}(\mathbb{F}_p)|$
$\rightsquigarrow$ Understand $H^*(\mathbf{X})$

Theorem: If $|\mathbf{X}(\mathbb{F}_p)| \in \mathbb{Z}[p]$
$\rightsquigarrow$ It's the E-poly. of $\mathbf{X} =: E(\mathbf{X})$

$$\dim(\mathbf{X}) = \deg(E(\mathbf{X}))$$
$$\chi(\mathbf{X}) = E(\mathbf{X})|_{p=1}$$
$$|\pi_0(\mathbf{X})| = \text{lead. coeff. of } E(\mathbf{X})$$

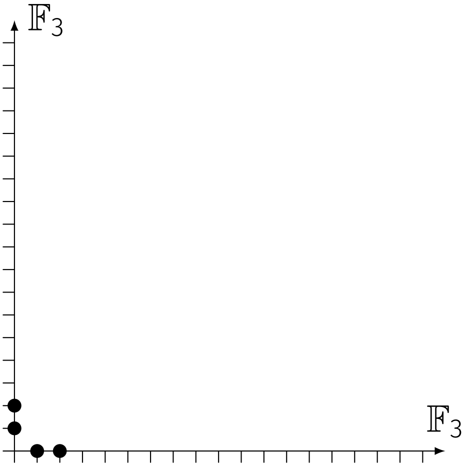
EXAMPLE: THE CIRCLE MOD $p = 2$

$$X\colon x^2 + y^2 = 1$$



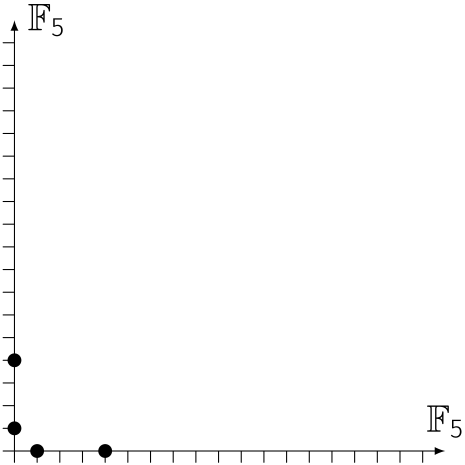
Example: The circle mod $p = 3$

$$X\colon x^2 + y^2 = 1$$



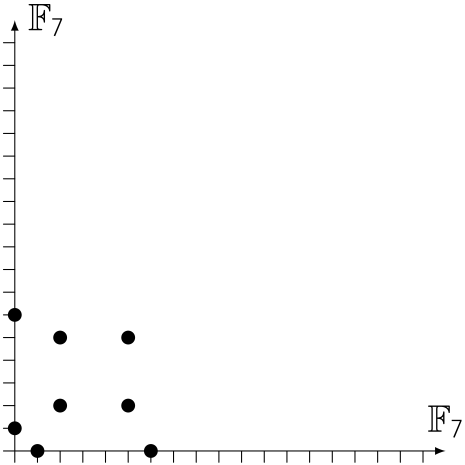
Example: The circle mod $p = 5$

$$X\colon x^2 + y^2 = 1$$



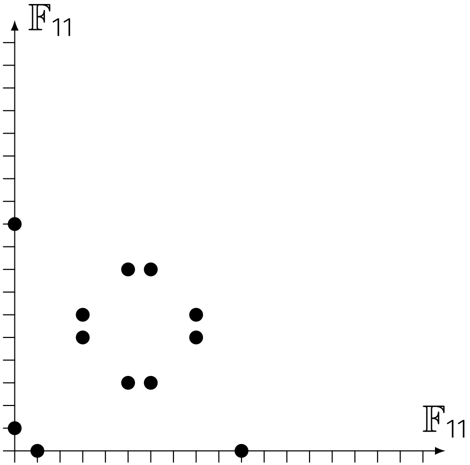
EXAMPLE: THE CIRCLE MOD $p = 7$

$$X\colon x^2 + y^2 = 1$$



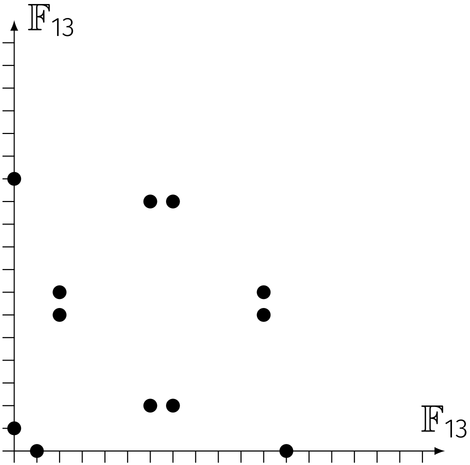
Example: The circle mod $p = 11$

$$X\colon x^2 + y^2 = 1$$



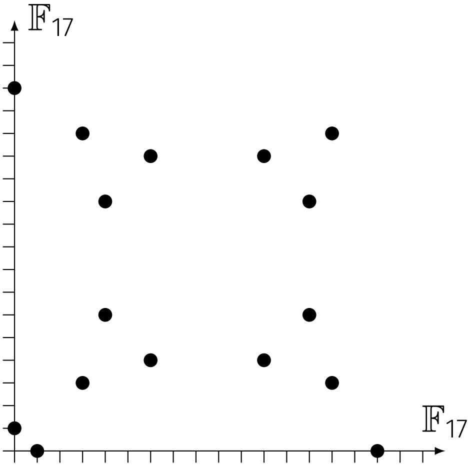
EXAMPLE: THE CIRCLE MOD $p = 13$

$$X\colon x^2 + y^2 = 1$$



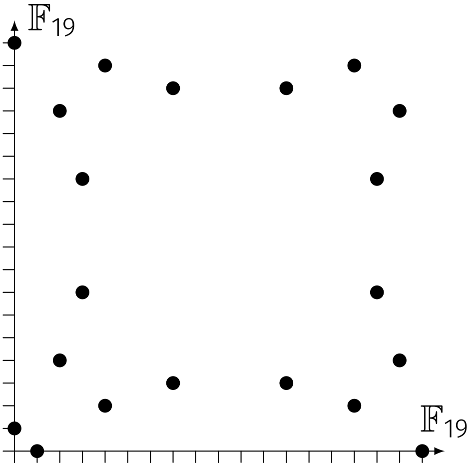
Example: The circle mod $p = 17$

$$X\colon x^2 + y^2 = 1$$



EXAMPLE: THE CIRCLE MOD $p = 19$

$$X\colon x^2 + y^2 = 1$$



Example: The circle mod p

$$\mathsf{X}\colon x^2 + y^2 = 1$$

$$|\mathsf{X}(\mathbb{F}_p)| = p - 1 = E(\mathsf{X})$$

$$\dim(\mathsf{X}) = \deg(E(\mathsf{X})) = 1$$

$$\chi(\mathsf{X}) = E(\mathsf{X})|_{p=1} = 0$$

$$|\pi_0(\mathsf{X})| = \text{lead. coeff. of } E(\mathsf{X}) = 1$$

How to count points?

Question: How do we count points on $\mathbf{R}$ and $\mathbf{R}/G$?

$$|\mathbf{R}(\mathbb{F}_p)| = \left| \left\{ \begin{array}{c} (A_1, \ldots B_g, Y_1, \ldots Y_r) \in G(\mathbb{F}_p) \text{ such that} \\ [A_1, B_1] \cdots [A_g, B_g]\, Y_1 \cdots Y_r = 1 \text{ and } Y_i \in C_i(\mathbb{F}_p) \end{array} \right\} \right|$$

Example: Surface S with genus $g = 1$ and $r = 3$ punctures:

$$|\mathbf{R}(\mathbb{F}_p)| = \left| \left\{ \begin{array}{c} (A, B, Y_1, Y_2, Y_3) \in G(\mathbb{F}_p) \text{ such that} \\ [A, B]\, Y_1 Y_2 Y_3 = 1 \text{ and } Y_i \in C_i(\mathbb{F}_p) \end{array} \right\} \right|$$

How to count points?

Answer: Frobenius says

$$\frac{|\mathbf{R}(\mathbb{F}_p)|}{|G(\mathbb{F}_p)|} = \sum_{\chi \in \mathrm{Rep}(G(\mathbb{F}_p))} \left(\frac{|G(\mathbb{F}_p)|}{\chi(1)} \right)^{2g-2} \prod_i \frac{\chi(C_i(\mathbb{F}_p))}{\chi(1)} |C_i(\mathbb{F}_p)|$$

Example: Surface S with genus $g = 1$ and $r = 3$ punctures:

$$|\mathbf{R}(\mathbb{F}_p)| \approx \sum_{\chi \in \mathrm{Rep}(G(\mathbb{F}_p))} \frac{\chi(C_1(\mathbb{F}_p))\,\chi(C_2(\mathbb{F}_p))\,\chi(C_3(\mathbb{F}_p))}{\chi(1)^3}$$

Goal: Compute $|\mathbf{R}(\mathbb{F}_p)|$ and show it is polynomial in p.

How to count points?

Problem: $\mathrm{Rep}(G(\mathbb{F}_p))$ is difficult to understand.

Series	Author(s)
$\mathrm{PSL}_2(p)$	Frobenius [Fro96] (p prime)
$\mathrm{SL}_2(q)$	Jordan [Jor07], Schur [Schu07] (see also [Bo11])
$\mathrm{GL}_2(q)$	Jordan [Jor07], Schur [Schu07], Steinberg [St51b]
$\mathrm{GL}_3(q)$, $\mathrm{GL}_4(q)$	Steinberg [St51b]
${}^2B_2(q^2)$	Suzuki [Suz62]
$\mathrm{GU}_3(q)$	Ennola [Enn63]
${}^2G_2(q^2)$	Ward [War66]
$\mathrm{Sp}_4(q)$	Srinivasan [Sr68] (q odd), Enomoto [Eno72] ($q = 2^m$)
$\mathrm{CSp}_4(q)$	Shinoda [Shi82] (q odd)
$\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$	Simpson and Frame [SiFr73]
$G_2(q)$	Chang and Ree [ChRe74] ($q = p^m$, $p \neq 2, 3$), see also Hiss [Hi90b, Anhang B], Enomoto [Eno76] ($q = 3^m$), Enomoto and Yamada [EnYa86] ($q = 2^m$)
$\mathrm{Sp}_6(q)$	Locker [Loc77] ($q = 2^m$), see also Lübeck [Lue93]
$\mathrm{CSp}_6(q)$	Lübeck [Lue93] (q odd)
${}^3D_4(q)$	Spaltenstein [Spa82b], Deriziotis and Michler [DeMi87]
${}^2F_4(q^2)$	Malle [Ma90] (complete table in CHEVIE [GHLMP])
$\mathrm{SO}_8^+(q)$ (partial)	Geck and Pfeiffer [GePf92] (q odd), Geck [Ge95] ($q = 2^m$)
$\mathrm{SO}_8^-(q)$ (partial)	Lübeck [GHLMP]

Annals of Mathematics, **103** (1976), 103-161

Representations of reductive groups over finite fields

By P. Deligne and G. Lusztig

Introduction

Let us consider a connected, reductive algebraic group G, defined over a finite field $\mathbf{F}_q$, with Frobenius map F. We shall be concerned with the representation theory of the finite group G^F, over fields of characteristic 0.

In 1968, Macdonald conjectured, on the basis of the character tables known at the time (GL_n, Sp_4), that there should be a well defined correspondence which, to any F-stable maximal torus T of G and a character θ of T^F in general position, associates an irreducible representation of G^F; moreover, if T modulo the centre of G is anisotropic over $\mathbf{F}_q$, the corresponding representation of G^F should be cuspidal (see Seminar on algebraic groups and related finite groups, by A. Borel et al., Lecture Notes in Mathematics, 131, pp. 117 and 101). In this paper we prove Macdonald's conjecture. More precisely, for T as above and θ an arbitrary character of T^F we construct virtual representations R_T^θ which have all the required properties.

Invent. math. 174, 555–624 (2008)
DOI: 10.1007/s00222-008-0142-x

Inventiones mathematicae

Mixed Hodge polynomials of character varieties

With an appendix by Nicholas M. Katz

Tamás Hausel[1], **Fernando Rodriguez-Villegas**[2]

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Abstract. We calculate the E-polynomials of certain twisted $\mathrm{GL}(n, \mathbb{C})$-character varieties $\mathcal{M}_n$ of Riemann surfaces by counting points over finite fields using the character table of the finite group of Lie-type $\mathrm{GL}(n, \mathbb{F}_q)$ and a theorem proved in the appendix by N. Katz. We deduce from this calculation several geometric results, for example, the value of the topological Euler characteristic of the associated $\mathrm{PGL}(n, \mathbb{C})$-character variety. The calculation also leads to several conjectures about the cohomology of $\mathcal{M}_n$: an explicit conjecture for its mixed Hodge polynomial; a conjectured curious hard Lefschetz theorem and a conjecture relating the pure part to absolutely indecomposable representations of a certain quiver. We prove these conjectures for $n = 2$.

ARITHMETIC HARMONIC ANALYSIS ON CHARACTER AND QUIVER VARIETIES

TAMÁS HAUSEL, EMMANUEL LETELLIER, and FERNANDO RODRIGUEZ-VILLEGAS

Abstract

We propose a general conjecture for the mixed Hodge polynomial of the generic character varieties of representations of the fundamental group of a Riemann surface of genus g to $\mathrm{GL}_n(\mathbb{C})$ *with fixed generic semisimple conjugacy classes at k punctures. This conjecture generalizes the Cauchy identity for Macdonald polynomials and is a common generalization of two formulas that we prove in this paper. The first is a formula for the E-polynomial of these character varieties which we obtain using the character table of* $\mathrm{GL}_n(\mathbb{F}_q)$*. We use this formula to compute the Euler characteristic of character varieties. The second formula gives the Poincaré polynomial of certain associated quiver varieties which we obtain using the character table of* $\mathfrak{gl}_n(\mathbb{F}_q)$*. In the last main result we prove that the Poincaré polynomials of the quiver varieties equal certain multiplicities in the tensor product of irreducible characters of* $\mathrm{GL}_n(\mathbb{F}_q)$*. As a consequence we find a curious connection between Kac-Moody algebras associated with comet-shaped, and typically wild, quivers and the representation theory of* $\mathrm{GL}_n(\mathbb{F}_q)$*.*

Setting: $G = \mathrm{GL}_n$

Theorem (Hausel–Letellier–Rodriguez-Villegas, '11)
$|(\mathbf{R}/G)(\mathbb{F}_p)|$ is a polynomial in p

- $\dim(\mathbf{R}/G) = (2g - 2 + r)\dim(\mathrm{GL}_n) - r \cdot \mathrm{rank}(\mathrm{GL}_n) + 2$
- $\chi(\mathbf{R}/G) = 0$ if $g > 0$
- $|\pi_0(\mathbf{R}/G)| = 1 \rightsquigarrow \mathbf{R}/G$ is connected

Phenomenom: $|(\mathbf{R}/G)(\mathbb{F}_p)|$ is a palindrome

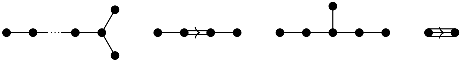
Sp_{2n}-Character varieties

Setting: $G = \mathrm{Sp}_{2n}$

Theorem (Cambó, '17)
$|(\mathbf{R}/G)(\mathbb{F}_p)|$ is a polynomial in p

- $\dim(\mathbf{R}/G) = (2g - 2 + r)\dim(\mathrm{Sp}_{2n}) - r \cdot \mathrm{rank}(\mathrm{Sp}_{2n})$
- $\chi(\mathbf{R}/G) = 0$ if $g > 1$
- $|\pi_0(\mathbf{R}/G)| = 1 \rightsquigarrow \mathbf{R}/G$ is connected

Phenomenom: $|(\mathbf{R}/G)(\mathbb{F}_p)|$ is a palindrome

G-Character varieties 

Setting: $G = (\text{alg. gp.}) + (\text{connected centre}) + (\text{reductive})$

Theorem (Kamgarpour–Nam–W.-Giannini, '24)
$|(\mathbf{R}/G)(\mathbb{F}_p)|$ is a polynomial in p

- $\dim(\mathbf{R}/G) = (2g - 2 + r)\dim G - r \cdot \operatorname{rank} G + 2\dim Z$
- $\chi(\mathbf{R}/G) = 0$ if $g > 1$ or $\dim Z > 0$
- $|\pi_0(\mathbf{R}/G)| = |\pi_0(Z(G^\vee))|$

Phenomenom: $|(\mathbf{R}/G)(\mathbb{F}_p)|$ is a palindrome

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julia> using CharacterVarieties

julia> G=rootdatum(:gl,2) # G=GL2
gl₂

julia> EX(G,1,3) # g=1 and r=3, i.e. torus with 3 punctures
Pol{Int64}: q⁸+q⁷-2q⁶-5q⁵+10q⁴-5q³-2q²+q+1

julia> G=rootdatum(:so,5) # G=SO5
so₅

julia> EX(G,1,3)
Pol{Int64}: 2q²⁴+12q²³+36q²²+76q²¹+126q²⁰+168q¹⁹+192q¹⁸+216q¹⁷+246q¹⁶+220q¹⁵-132q¹⁴
+588q¹³+1108q¹²+588q¹¹-132q¹⁰+220q⁹+246q⁸+216q⁷+192q⁶+168q⁵+126q⁴+76q³+36q²+12q+2
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