Character varieties in arbitrary type

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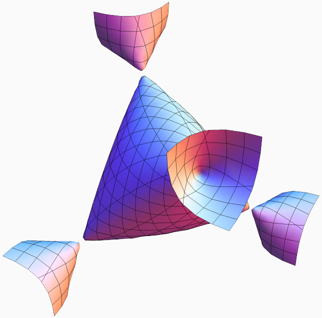


The spaces I study

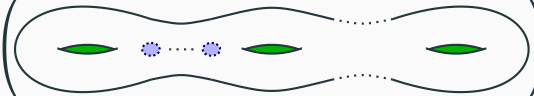
Data: algebraic group G, and f.g. group Γ.

$$G,\ \Gamma \rightsquigarrow \mathrm{Hom}(\Gamma, G) \curvearrowleft G \text{ by conj.}$$

If $\Gamma = \pi_1(\text{Surface})$, we are in the world of the Riemann–Hilbert correspondence, non-abelian Hodge theory, the Langlands program, mirror symmetry, etc.

The spaces I study

Surface S with genus $g \geq 0$ and $r \geq 0$ punctures.

$\pi_1(S) = \pi_1\Big($  $\Big)$

$$= \frac{\langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_r \rangle}{[a_1, b_1] \cdots [a_g, b_g] y_1 \cdots y_r}$$

Fix conj. classes $C_1, \ldots, C_r \subseteq G$. The representation variety is

$$\mathbf{R} := \left\{ f \colon \pi_1(S) \to G \;\middle|\; f(y_i) \in C_i \right\} \curvearrowleft G \text{ by conj.}$$

$$\rightsquigarrow [\mathbf{R}/G] \quad \text{and} \quad \mathbf{R} /\!\!/ G$$

How to study a space?

NUMBERS OF SOLUTIONS OF EQUATIONS IN FINITE FIELDS

ANDRÉ WEIL

The equations to be considered here are those of the type

$$a_0 x_0^{n_0} + a_1 x_1^{n_1} + \cdots + a_r x_r^{n_r} = b. \tag{1}$$

Such equations have an interesting history. In art. 358 of the *Disquisitiones* [**1 a**],[1] Gauss determines the Gaussian sums (the so-called cyclotomic "periods") of order 3, for a prime of the form $p=3n+1$, and at the same time obtains the numbers of solutions for all congruences $ax^3-by^3\equiv 1 \pmod{p}$. He draws attention himself to the elegance of his method, as well as to its wide scope; it is only much later, however, viz. in his first memoir on biquadratic residues [**1b**], that he gave in print another application of the same method; there he treats the next higher case, finds the number of solutions of any congruence $ax^4-by^4\equiv 1 \pmod{p}$, for a prime of the form $p=4n+1$, and derives from this the biquadratic character of 2 mod p, this being the ostensible purpose of the whole highly ingenious and intricate investigation. As an incidental consequence ("*coronidis loco*," p. 89), he also gives in substance the number of solutions of any congruence $y^2\equiv ax^4-b \pmod{p}$; this result includes as a special case the theorem stated as a conjecture ("*observatio per inductionem facta gravissima*") in the last entry of his *Tagebuch* [**1c**];[2] and it implies the truth of what has lately become known as the Riemann hypothesis, for the function-field defined by that equation over the prime field of p elements.

Gauss' procedure is wholly elementary, and makes no use of the Gaussian sums, since it is rather his purpose to apply it to the determination of such sums. If one tries to apply it to more general cases, however, calculations soon become unwieldy, and one realizes the necessity of inverting it by taking Gaussian sums as a starting point. The means for doing so were supplied, as early as 1827, by Jacobi, in a letter to Gauss [**2a**] (cf. [**2b**]). But Lebesgue, who in 1837 devoted two papers [**3a, b**] to the case $n_0 = \cdots = n_r$ of equation (1), did not

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] It is surprising that this should have been overlooked by Dedekind and other authors who have discussed that conjecture (cf. M. Deuring, Abh. Math. Sem. Hamburgischen Univ. vol. 14 (1941) pp. 197–198).

Fix X a variety over $\mathbb{Z}$.

Moral [Weil]

Count $|\mathsf{X}(\mathbb{F}_q)|$

$\rightsquigarrow$ Understand $H^*(\mathsf{X})$

Theorem [Katz]

The polynomial $q \mapsto |\mathsf{X}(\mathbb{F}_q)|$ is an invariant of $H^*(\mathsf{X})$

How to study a space?

Example: Consider $\mathbb{G}_{\mathrm{m}}$. This is defined by $xy - 1 = 0$. Then

$$|\mathbb{G}_{\mathrm{m}}(\mathbb{F}_q)| = |\{(x, y) \in \mathbb{F}_q^2 \mid xy - 1 = 0\}| = q - 1.$$

We read $q - 1$ in the following way:

$$\dim \mathbb{G}_{\mathrm{m}} = \text{degree} = 1$$

$$\chi(\mathbb{G}_{\mathrm{m}}) = (q-1)\Big|_{q \mapsto 1} = 0$$

$$\# \text{ of components} = \text{leading coeff.} = 1$$

How to count points?

Question: How do we count points on R and $\mathsf{R}/\!\!/G$?

Answer:

$$\frac{|\mathsf{R}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \sum_{\chi \in \mathsf{Irr}(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-2} \prod_i \frac{\chi(C_i(\mathbb{F}_q))}{\chi(1)} |C_i(\mathbb{F}_q)|.$$

Goal: Compute this expression and show it is polynomial in q.

Problems:

(1) $\mathsf{Irr}(G(\mathbb{F}_q))$ is hard to understand.

(2) $\mathsf{Irr}(G(\mathbb{F}_q))$ depends on q.

(3) $\chi(C_i(\mathbb{F}_q))$ is hard to evaluate.

(4) Bridge from $|\mathsf{R}(\mathbb{F}_q)|$ to $|(\mathsf{R}/\!\!/G)(\mathbb{F}_q)|$ is unclear.

How to count points?

Solutions:

(1) $\mathrm{Irr}(G(\mathbb{F}_q))$ is hard to understand.

Fix G reductive $\rightsquigarrow$ Deligne–Lusztig theory.

(2) $\mathrm{Irr}(G(\mathbb{F}_q))$ depends on q.

Reparameterise sum over data independent of q.

(3) $\chi(C_i(\mathbb{F}_q))$ is hard to evaluate.

Pick C_i semisimple and regular.

(4) Bridge from $|\mathbf{R}(\mathbb{F}_q)|$ to $|(\mathbf{R}/\!\!/G)(\mathbb{F}_q)|$ is unclear.

Pick C_i generically.

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Inventiones mathematicae

Mixed Hodge polynomials of character varieties

With an appendix by Nicholas M. Katz

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Abstract. We calculate the E-polynomials of certain twisted $\mathrm{GL}(n,\mathbb{C})$-character varieties $\mathcal{M}_n$ of Riemann surfaces by counting points over finite fields using the character table of the finite group of Lie-type $\mathrm{GL}(n,\mathbb{F}_q)$ and a theorem proved in the appendix by N. Katz. We deduce from this calculation several geometric results, for example, the value of the topological Euler characteristic of the associated $\mathrm{PGL}(n,\mathbb{C})$-character variety. The calculation also leads to several conjectures about the cohomology of $\mathcal{M}_n$: an explicit conjecture for its mixed Hodge polynomial; a conjectured curious hard Lefschetz theorem and a conjecture relating the pure part to absolutely indecomposable representations of a certain quiver. We prove these conjectures for $n = 2$.

Contents

1 Introduction

Let $g \geq 0$ and $n > 0$ be integers. Let $\zeta_n \in \mathbb{C}$ be a primitive n-th root of unity. Abbreviating $[A, B] = ABA^{-1}B^{-1}$ and denoting the identity matrix

ARITHMETIC HARMONIC ANALYSIS ON CHARACTER AND QUIVER VARIETIES

TAMÁS HAUSEL, EMMANUEL LETELLIER, and FERNANDO RODRIGUEZ-VILLEGAS

Abstract
We propose a general conjecture for the mixed Hodge polynomial of the generic character varieties of representations of the fundamental group of a Riemann surface of genus g to $\mathrm{GL}_n(\mathbb{C})$ with fixed generic semisimple conjugacy classes at k punctures. This conjecture generalizes the Cauchy identity for Macdonald polynomials and is a common generalization of two formulas that we prove in this paper. The first is a formula for the E-polynomial of these character varieties which we obtain using the character table of $\mathrm{GL}_n(\mathbb{F}_q)$. We use this formula to compute the Euler characteristic of character varieties. The second formula gives the Poincaré polynomial of certain associated quiver varieties which we obtain using the character table of $\mathfrak{gl}_n(\mathbb{F}_q)$. In the last main result we prove that the Poincaré polynomials of the quiver varieties equal certain multiplicities in the tensor product of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$. As a consequence we find a curious connection between Kac-Moody algebras associated with comet-shaped, and typically wild, quivers and the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$.

Contents

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Setting:

- $G = \mathrm{GL}_n$
- $C_1, \ldots, C_r$ reg. s.s. generic conj. classes

Theorem (Hausel–Letellier–Rodriguez-Villegas, '11)

$|(\mathbf{R}/\!\!/G)(\mathbb{F}_q)|$ is a polynomial in q and we have

- $\dim \mathbf{R}/\!\!/G = (2g - 2 + r) \cdot n^2 + 2 - r \cdot n$
- $\chi(\mathbf{R}/\!\!/G) = 0$ if $g > 0$
- $|\pi_0(\mathbf{R}/\!\!/G)| = 1 \rightsquigarrow \mathbf{R}/\!\!/G$ is connected

Phenomenom: $|(\mathbf{R}/\!\!/G)(\mathbb{F}_q)|$ is a palindrome

Character varieties in arbitrary type

Setting:

- G conn. split red. group with conn. centre $Z(G)$
- $C_1, \ldots, C_r$ reg. s.s. generic conj. classes

Theorem (Kamgarpour–Nam–W., '23)

$|(\mathbf{R}/\!/G)(\mathbb{F}_q)|$ is a polynomial in q and we have

- $\dim \mathbf{R}/\!/G = (2g - 2 + r)\dim G + 2\dim Z(G) - r \cdot \operatorname{rank} G$
- $\chi(\mathbf{R}/\!/G) = 0$ if $g > 1$ or $\dim Z > 0$
- $|\pi_0(\mathbf{R}/\!/G)| = |\pi_0(Z(G^\vee))|$ if $g > 0$ or $r > 3$

Same phenomenom: $|(\mathbf{R}/\!/G)(\mathbb{F}_q)|$ is a palindrome

Polynomials in the variable q and t

$$|(\mathbf{R}/\!\!/G)(\mathbb{F}_q)| \xleftarrow{\;t=-1\;} \underset{H(\mathbf{R}/\!\!/G;q,t)}{\text{Mixed Hodge poly.}} \dashrightarrow |\mathbf{Y}(\mathbb{F}_q)|$$

When $G = \mathrm{GL}_2$:

$\lvert(\mathbf{R}/\!\!/G)(\mathbb{F}_q)\rvert$	$H(\mathbf{R}/\!\!/G;q,t)$	$\lvert\mathbf{Y}(\mathbb{F}_q)\rvert$
$= q^{2g-1}(q-1)^{4g-2}(q+1)^{2g-1}$	$= \frac{(qt^4)^{2g-1}(1+qt)^{2g-1}(1+q^2t^3)^{2g}}{(1-qt)(1-qt^2)}$	$= \frac{q^{8g-3}}{q-1}$
$+ (q-1)^{4g-2}(q+1)^{2g+1}$	$+ \frac{t^{8g-4}(1+qt)^{2g-1}(1+q^2t)^{2g}}{(1-q)(1-qt)}$	$+ q^{6g-3}$
$+ -q^{2g-1}(q-1)^{4g-2}$	$+ \frac{(qt^4)^{2g-1}(1+qt)^{4g}}{(1-q)(1-qt^2)}$	$+ -\frac{q^{6g-2}}{q-1}$
$+ 0$	$+ 0$	$+ 0$

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