The Representation Variety and its E-Polynomial

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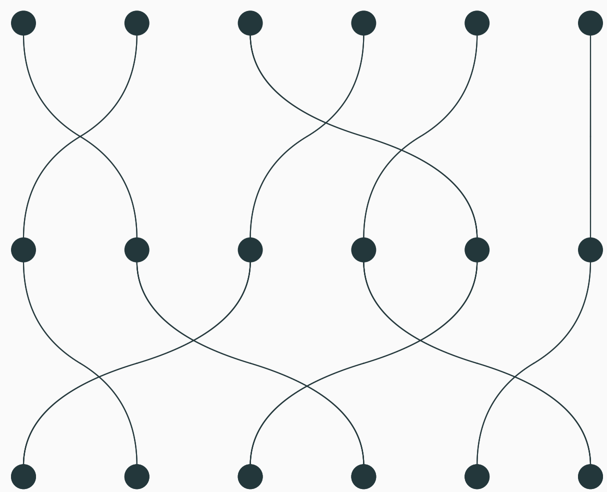

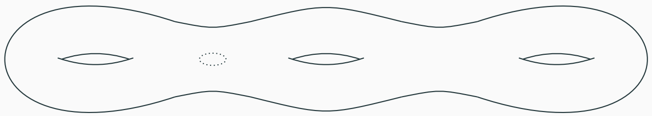

Figure 1: The once-punctured genus 3 compact orientable surface.

- $\Sigma_g :=$ once-punctured genus g compact orientable surface with fundamental group

$$\Gamma_g := \pi_1(\Sigma_g) = \frac{\langle a_1, b_1, \dots, a_g, b_g, z \rangle}{[a_1, b_1] \dots [a_g, b_g] z}.$$

- $G =$ reductive group (think GL_n or a 'nice' subgroup).
- $C = [s] =$ semisimple regular split conjugacy class of G (think $s = \mathrm{diag}(s_1, \dots, s_n)$ with $s_i \neq s_j$ for $i \neq j$).

The representation variety associated to this data is

$$\mathbf{R} := \left\{ (A_1, B_1, \dots, A_g, B_g, Z) \in G^{2g} \times C \;\middle|\; [A_1, B_1] \dots [A_g, B_g] Z = 1 \right\}.$$

We want to understand the topology of the representation variety.

In particular, we seek an expression for the *E-polynomial* of **R**, which contains desired topological data, e.g. Euler characteristic, dimension, number of connected components.

We will do this by computing $|\mathbf{R}(\mathbb{F}_q)|$, the number of $\mathbb{F}_q$-points of **R**:

$$\mathbf{R}(\mathbb{F}_q) := \left\{ \begin{matrix} A_1, B_1, \dots, A_g, B_g \in G(\mathbb{F}_q)^{2g}, \\ Z \in C(\mathbb{F}_q) \end{matrix} \;\middle|\; [A_1, B_1] \dots [A_g, B_g] Z = 1 \right\}.$$

Theorem [Katz]

If $\mathbf{X}$ is an algebraic variety and $P_{\mathbf{X}} \in \mathbb{Z}[x]$ is a polynomial such that $|\mathbf{X}(\mathbb{F}_q)| = P_{\mathbf{X}}(q)$ then $P_{\mathbf{X}}$ is the E-polynomial of $\mathbf{X}$.

Then computing the E-polynomial reduces to the problem of showing $|\mathbf{R}(\mathbb{F}_q)|$ is a polynomial in q, and explicitly computing this polynomial.

$$|\mathsf{R}(\mathbb{F}_q)|$$

↓ Frobenius mass formula

$$|G(\mathbb{F}_q)| \sum_{\chi \in \widehat{G(\mathbb{F}_q)}} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-1} \chi(s)$$

→ Formulas of Hausel–Letellier–Villegas ($G = \mathrm{GL}_n$) →

$$\mathbb{H}_\mu(q^{1/2}, q^{-1/2})$$

↓ Formulas of Deligne–Lusztig (G reductive)

$$|G(\mathbb{F}_q)| \sum_{\substack{L \subseteq W \\ \text{refl. subgp}}} \sum_{\substack{\theta \in \widehat{T(\mathbb{F}_q)} \\ W_\theta = L}} \sum_{\chi \in \mathcal{B}(\theta)} \langle \chi, \mathcal{B}(\theta) \rangle \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-1} \chi(s)$$

⇢ $G = \mathrm{GL}_n$ ⇢ $\mathbb{H}_\mu(q^{1/2}, q^{-1/2})$

⇢ Conjecture for $G =$ reductive ⇢ Hodge polynomial $H(\mathsf{R}; x, y, t)$

$\mathbb{H}_\mu(q^{1/2}, q^{-1/2})$ ⇝ Conjecture for $G = \mathrm{GL}_n$ (HLV) ⇝ Hodge polynomial $H(\mathsf{R}; x, y, t)$

References

- *Mixed Hodge polynomials of character varieties, with an appendix by Nicholas M. Katz*, Tamas Hausel, Fernando Rodriguez-Villegas, `https://arxiv.org/pdf/math/0612668.pdf`, 2008.
- *Arithmetic harmonic analysis on character and quiver varieties*, Tamas Hausel, Emmanuel Letellier, Fernando Rodriguez-Villegas, `https://arxiv.org/pdf/0810.2076.pdf`, 2011.
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