Kinky Sets

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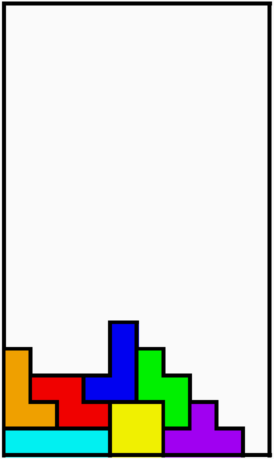
The Game

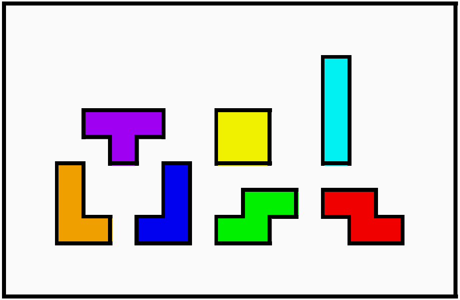
A typical game of Tetris.

The playing field is 10 blocks wide and 20 blocks high.



The Pieces

There are 7 pieces in Tetris, they are 7 distinct shapes that can be made with 4 blocks, up to rotation.



They're called the T piece, O piece, I piece, L piece, J piece, S piece and Z piece.

The Goal

The aim of the game:

- Clear lines by filling them up with pieces.
- Lines clear once they're filled.
- You score points for each line.

WE DON'T CARE ABOUT THIS

We are only concerned with one question:

Can I play Tetris forever?

The Assumption

Normally, the game speeds up and you must react faster.
Every player will, eventually, die because of this speed.

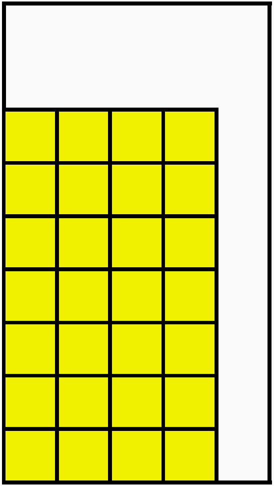
Thus, we impose an assumption:

The game plays as slowly as we want. Then we can always place any piece anywhere we want.

Some Simple Cases

The question is obvious when we simplify the game.

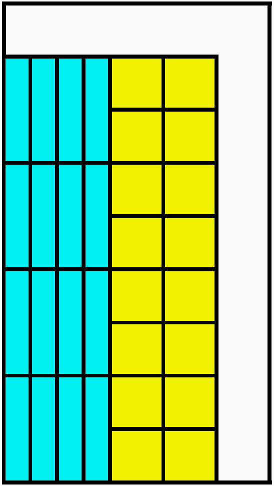
If the only piece were the O piece then I could clearly stack them neatly.



Some Simple Cases

The question is obvious when we simplify the game.

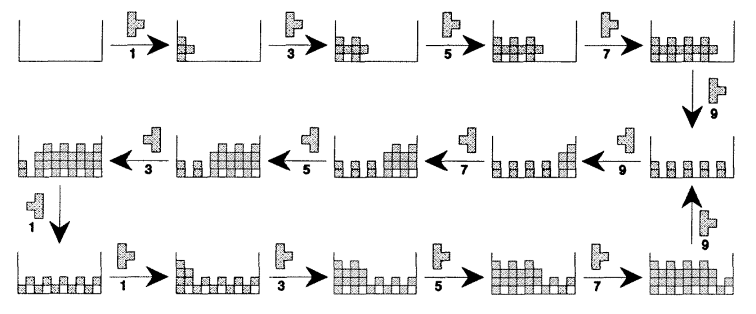
If the only pieces were the O piece and the I piece then I could clearly stack them neatly as well.



Some Simple Cases

The question is obvious when we simplify the game.

If the only piece were the T piece then I could stack them neatly, but it is less clear how to do this.

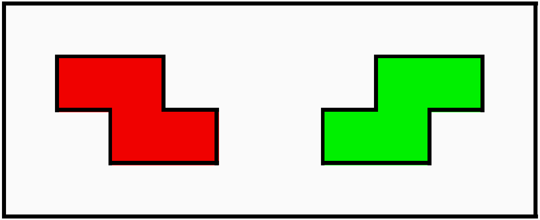


Some Simple Cases

Similar strategies exist for all *small* combinations of the O piece, I piece, T piece, L piece, and J piece (See [T] Thm 2).

However, there are two problem pieces:

the *Z piece* and the *S piece*.



Kinky Sets

A *kinky set* is a set of pieces consisting of only the Z piece and S piece, alternating in orientation. That is, a sequence of the form

...ZSZSZSZSZSZSZSZSZSZS...

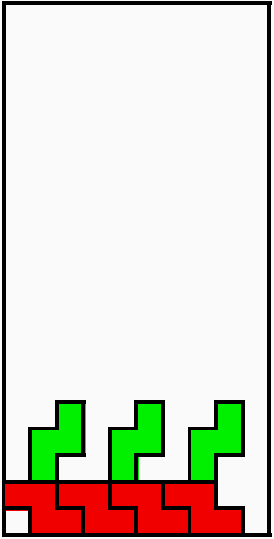
We will now see that a set of Kinks like this of sufficiently large size will always end our game.

Kinky Sets

Lemma 1 (Burgiel)

Consider a Tetris game where only alternating Z pieces and S pieces are presented to the player.

No more than 120 Kinks can be played vertically, with their leftmost cells in an even column, or horizontally, in any column, without losing.

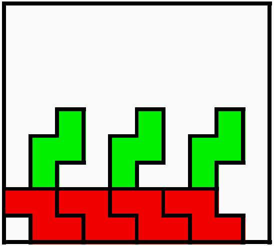


Kinky Sets

Proof

Number the columns of the playing field from 1 to 10, left to right. Let

- $b_i = \#$ of cells that are filled with a Tetris block in column i.
- $h_i = \#$ of horizontal Kinks in the columns $i-1$, i, and $i+1$.
- $v_i = \#$ of vertical Kinks in columns i and $i+1$.



For example,

$b_1 = 1,\ b_2 = 4,\ b_3 = 4,\ \ldots$

$h_1 = 0,\ h_2 = 2,\ h_3 = 2,\ \ldots$

$v_1 = 0,\ v_1 = 1,\ v_2 = 1,\ v_3 = 1,\ \ldots$

Kinky Sets

Proof

We notice that

$$b_i = \underbrace{2v_{i-1} + 2v_i}_{\text{vertical pieces}} + \underbrace{h_{i-1} + 2h_i + h_{i+1}}_{\text{horizontal pieces}}.$$

If $b_i - b_j > 20$ then we're above the playing field height, so we've lost. Thus we must have $b_i - b_j \leq 20$.

In addition, we cannot play Kinks outside of the playing field, so $h_1 = h_{10} = 0$. Then the above tells us that

$$b_2 - b_1 = 2v_2 + h_2 + h_3 \leq 20. \tag{1}$$

Similarly, $2v_8 + h_8 + h_9 \leq 20$.

Kinky Sets

Proof

More generally, we have

$$b_{i+1} - b_i = 2v_{i+1} - 2v_{i-1} + h_{i+1} + h_{i+2} - h_i - h_{i-1} \leq 20.$$

We rearrange this to see that

$$2v_{i+1} + h_{i+1} + h_{i+2} \leq 20 + 2v_{i-1} + h_{i-1} + h_i. \tag{2}$$

We let $i = 3$ in (2) and substitute (1) into (2) to find that $2v_4 + h_4 + h_5 \leq 40$. Similarly, we find that $2v_6 + h_6 + h_7 \leq 40$. Also notice that $2v_{10} + h_{10} + h_{11} = 0$. We conclude that

$$\underbrace{\sum_{i=1}^{10} h_i}_{\text{\# of horizontal pieces}} + \underbrace{\sum_{i=1}^{5} v_{2i}}_{\text{\# of vertical pieces}} \leq 120.$$

Kinky Sets

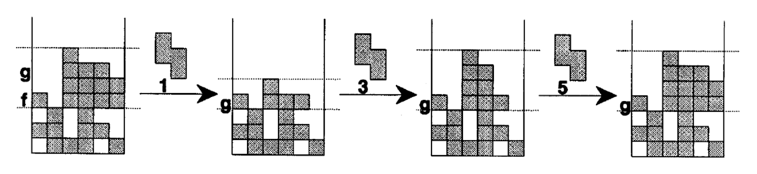
Lemma 2 (Burgiel)

A Tetris game involving alternating Z pieces and S pieces will always end before 70000 of these pieces are played.

Proof

Lemma 1 tells us that we must play the Kinks vertically with their leftmost pieces in odd-numbered columns. If we do not do this, Lemma 1 tells us that we will lose.

Roughly, this playstyle will always result in unfillable gaps.



Proof

Lemma 1 showed that at most 120 moves can be made with vertical Kinks, between columns, or horizontal Kinks, in any column, before a game is lost. Hence, a player can make 120 moves that fill at most two holes per move.

During one game a player might see as many as $50 + (120 \cdot 2) = 290$ holes. Since at most 240 Kinks can be played without forming a hole, it is only possible to play a total of $240 \cdot 290 = 69600$ pieces before losing.

Thus a game of alternating Kinks will always be lost before 70000 Kinks are played. $\square$

Probability

We can model the random piece generation of Tetris with a *Markov chain*.

This is a matrix, where the entry p_{ij} is the probability that the future value is i, given that the current value is j.

$$P = \begin{pmatrix} \frac{1}{28} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} \\ \frac{9}{56} & \frac{1}{28} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} \\ \frac{9}{56} & \frac{9}{56} & \frac{1}{28} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} \\ \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{1}{28} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} \\ \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{1}{28} & \frac{9}{56} & \frac{9}{56} \\ \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{1}{28} & \frac{9}{56} \\ \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{9}{56} & \frac{1}{28} \end{pmatrix}$$

(See STAT2003, STAT3004)

Probability

We can further model the random piece generation as a *random variable* on a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Here,

- The *state space* is $\Omega = \{T, O, I, L, J, S, Z\}$.
- The *sigma-field* is $\mathcal{F} = \mathcal{P}(\Omega) =$ the *powerset* of Ω.

$$\mathcal{P}(\Omega) = \{\{T\}, \{O\}, \{I\}, \{L\}, \{J\}, \{S\}, \{Z\}, \{T, O\}, \ldots\}.$$

- The *probability measure* is determined by the Markov chain.

(See STAT3004, STAT4404, MATH4405)

Conclusion

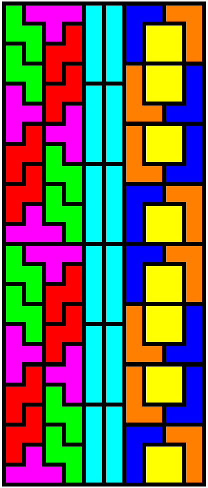
The Markov chain tells me that I will eventually see a string of 70000 Kinks. Thus, I will always die.

A Perfect Game

In recent versions of Tetris, the random piece generation uses a *bag system*.

The game draws new pieces from a 'bag' of 7 pieces. Once the bag is empty, we begin a new bag. This guarentees that we will not be killed by the Kinks.

In fact, there exists a perfect strategy.



References

T Tsuruda, K. (2010). *A Closer Look at Tetris: Analysis of a Variant Game*, Atlantic Electronic Journal of Mathematics Volume **4**, Number **1**, Winter **2010**. Department of Mathematics and Computing Science, Saint Mary's University, Canada.

Bu Burgiel, H. (1997). *How to lose at Tetris*, The Mathematical Gazette, 81(491), pp194-200. `doi:10.2307/3619195`.

Br Brzustowski, J. (1992). *Can you win at Tetris?*, Doctoral dissertation, University of British Columbia.