Hecke Algebras and Gelfand Pairs in Representation Theory

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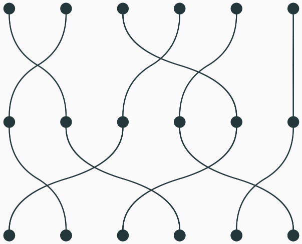


Contents

Historical Group Theory

Since antiquity, mathematicians have been concerned with solving polynomial equations.

- Degree $n = 1$, $ax + b = 0$: Clear
- Degree $n = 2$, $ax^2 + bx + c = 0$: Quadratic equation
- Degree $n = 3$, $ax^3 + bx^2 + cx + d = 0$: Cardano's formula
- Degree $n = 4$, $ax^4 + bx^3 + cx^2 + dx + e = 0$: Ferrari's method

What about degree $n \geq 5$?

In the early 19th century, Evariste Galois developed fundamental concepts of group theory to answer this question.

Groups and symmetry

- Galois understood that the roots of a polynomial equation possessed a certain symmetry.
- He came up with the idea of a Galois group to describe the symmetry of the roots.
- Soon after, the modern definition of a group was established.

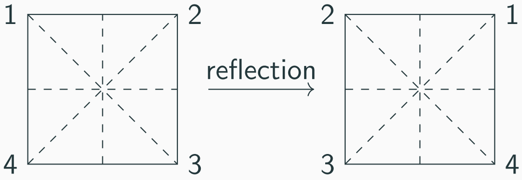
Examples of groups are:

$$(S_n, \circ), \quad (\mathbb{Z}_n, +), \quad (\mathbb{Z}, +), \quad (\mathbb{R}^+, \times),$$

$$(\mathrm{GL}_n(\mathbb{R}), \times), \quad (\mathrm{O}_n(\mathbb{R}), \times).$$

Groups in the real world

Some geometric examples of groups are the dihedral groups. The dihedral group D_{2m} is the set of $2m$ symmetries associated to the m-gon, with the group operation being composition of symmetries.



The dihedral group's action on the m-gon serves as example of a group acting on a geometric object.

This gives us an intuitive understanding of groups: they encode the symmetries of physical and mathematical objects.

Representations of groups

Definition

Consider a group G and a vector space V. We say that $\rho \colon G \to \mathrm{GL}(V)$ is a representation of G on V if ρ is a homomorphism of groups.

- The elements $\rho(g)$ are linear maps on V.
- Homomorphisms preserve the structure of G.
- We can study these elements with linear algebra.

For example, let $G = S_3$, and $V = \mathbb{C}^3$. Then $\mathrm{GL}(V) = \mathrm{GL}_3(\mathbb{C})$.

If $s_1 = (1\ 2)$ and $s_2 = (2\ 3)$, then we may write

$$\rho(s_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Constructing representations

- If K is a subgroup of a finite group G, then we can use representations of K to build representations of G via a procedure called induction.
- The representation of K that always exists is the trivial representation, i.e. the homomorphism $\mathbf{1}\colon K \to \mathbb{C}^\times$ given by $\mathbf{1}(k) = 1 \in \mathbb{C}$.
- We denote the induced representation by $\operatorname{Ind}_K^G \mathbf{1}$.

Irreducible representations as building blocks

We like to decompose representations into their irreducible components. These irreducible representations are the building blocks of all other representations.

Mashcke's theorem says, since $\operatorname{Ind}_K^G \mathbf{1}$ is a complex representation of the finite group G, then

$$\operatorname{Ind}_K^G \mathbf{1} = \bigoplus_{i=1}^{n} V_i,$$

where each V_i is irreducible. We say that $\operatorname{Ind}_K^G \mathbf{1}$ is multiplicity-free if $V_i \ncong V_j$ for each $i \neq j$.

To investigate the induced representation $\operatorname{Ind}_K^G \mathbf{1}$, we need the Hecke algebra.

The Hecke algebra

$\mathcal{H}(G, K)$ is the space of complex-valued functions on G that are K-bi-invariant. Explicitly,

$$\mathcal{H}(G, K) := \{f \colon G \to \mathbb{C} \mid f(kgk') = f(g)\ \forall g \in G, k, k' \in K\}.$$

This forms an algebra under the convolution product

$$(f \star f')(g) := \sum_{xy=g} f(x) f'(y) = \sum_{x \in G} f(gx) f'(x^{-1}).$$

Proposition

$\mathrm{Ind}_K^G \mathbf{1}$ is multiplicity-free if and only if $\mathcal{H}(G, K)$ is commutative.

Gelfand pairs

We call a pair of groups (G, K) is a Gelfand pair if the induced representation $\mathrm{Ind}_K^G \mathbf{1}$ is multiplicity free.

Examples of Gelfand pairs:

- (G, K) with G abelian,
- $(G \times G, G)$,
- $(\mathrm{O}_{n+1}(\mathbb{F}_q), \mathrm{O}_n(\mathbb{F}_q))$ with $q \neq 2^k$, and
- $(S_{m+n}, S_m \times S_n)$.

To prove that these are Gelfand pairs, we can use the following theorem:

Gelfand pairs

Theorem (Gelfand's Trick)

Let $\varphi\colon G \to G$ be a map such that

(i) $\varphi(ab) = \varphi(b)\varphi(a)$,

(ii) φ is a bijection,

(iii) $\varphi^2 = \mathrm{Id}_G$, and

(iv) $K\varphi(x)K = KxK$ for all $x \in G$.

Then $\mathcal{H}(G, K)$ is commutative.

We often consider $\varphi(x) = x^{-1}$, and for matrix groups, $\varphi(x) = x^t$.

A non-commutative Hecke algebra

Fix $G = \mathrm{GL}_2(\mathbb{F}_q)$ and $B = B(\mathbb{F}_q)$, the subgroup of upper-triangular matrices in G.

G has the Bruhat decomposition

$$G = \bigsqcup_{w \in S_2} BwB = B \sqcup BsB = \begin{pmatrix} \mathbb{F}_q & \mathbb{F}_q \\ 0 & \mathbb{F}_q \end{pmatrix} \sqcup \begin{pmatrix} \mathbb{F}_q & \mathbb{F}_q \\ \mathbb{F}_q^\times & \mathbb{F}_q \end{pmatrix}.$$

The Hecke algebra $\mathcal{H}(G, B)$ has a basis $\{\chi_B, \chi_{BsB}\}$.

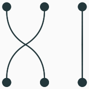
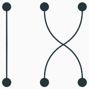
Set $I := \chi_B$ and $T := \chi_{BsB}$. We find that $\mathcal{H}(G, B)$ has the following presentation

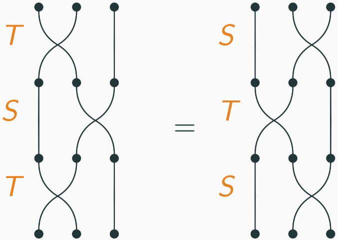
$$\mathcal{H}(G, B) = \langle T \mid T^2 = (q-1)T + qI \rangle.$$

A non-commutative Hecke algebra

If $G = \mathrm{GL}_3(\mathbb{F}_q)$ and $B = B(\mathbb{F}_q)$, we find that $\mathcal{H}(G, B)$ has the presentation

$$\mathcal{H}(G,B) = \left\langle T, S \;\middle|\; \begin{array}{l} T^2 = (q-1)T + qI, \\ S^2 = (q-1)S + qI, \end{array} \; TST = STS \right\rangle.$$

Associate T to  and S to . Then



A non-commutative Hecke algebra

If $G = \mathrm{GL}_4(\mathbb{F}_q)$ and $B = B(\mathbb{F}_q)$, we find that $\mathcal{H}(G, B)$ has the presentation

$$\mathcal{H}(G,B) = \left\langle T_1, T_2, T_3 \;\middle|\; \begin{array}{c} T_i^2 = (q-1)T_i + qI, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_1 T_3 = T_3 T_1 \end{array} \right\rangle.$$

We recall a standard presentation of S_4,

$$S_4 = \left\langle s_1, s_2, s_3 \;\middle|\; \begin{array}{c} s_i^2 = 1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ s_1 s_3 = s_3 s_1 \end{array} \right\rangle.$$

S_n is actually the *Weyl group* of GL_n!

A non-commutative Hecke algebra

S_n has the associated Coxeter matrix $M = (m_{ij})$ given by

$$m_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 3, & \text{if } |i-j| = 1, \\ 2, & \text{else.} \end{cases}$$

Then S_n has the Coxeter presentation

$$\left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{c} s_i^2 = 1, \\ \underbrace{s_i s_j s_i \ldots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij} \text{ terms}} \end{array} \right\rangle.$$

If $G = \mathrm{GL}_n(\mathbb{F}_q)$ and $B = B(\mathbb{F}_q)$ then $\mathcal{H}(G, B)$ has the presentation

$$\mathcal{H}(G,B) = \left\langle T_1, \ldots, T_{n-1} \;\middle|\; \begin{array}{c} T_i^2 = (q-1)T_i + qI, \\ \underbrace{T_i T_j T_i \ldots}_{m_{ij} \text{ terms}} = \underbrace{T_j T_i T_j \ldots}_{m_{ij} \text{ terms}} \end{array} \right\rangle.$$

Generalising $\mathcal{H}$ to non-finite groups

We can weaken our condition that G is finite to the condition that G is a locally compact topological group. For instance, $G = \mathbb{R}^n$, $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(F)$, for a non-archemedian local field F.

Take an open and compact subgroup K of G. Then the Hecke algebra is the space

$$C_c(K \backslash G / K) := \{f \colon G \to \mathbb{C} \mid f(kgk') = f(g),\ \operatorname{supp} f \text{ is compact}\}.$$

This forms an algebra under the convolution product

$$(f \star f')(x) = \int_G f(xg) f'(g^{-1}) \, \mathrm{d}\mu(g).$$

Further research: the spherical Hecke algebra

Fix a non-archemedian local field F (e.g. $F = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$).

Associated to F is its ring of integers $\mathcal{O}$ (e.g. if $F = \mathbb{Q}_p$ then $\mathcal{O} = \mathbb{Z}_p$, if $F = \mathbb{F}_q((t))$ then $\mathcal{O} = \mathbb{F}_q[[t]]$).

Consider $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathcal{O})$. Then the spherical Hecke algebra is $C_c(K\backslash G/K)$.

Theorem

The spherical Hecke algebra $C_c(K\backslash G/K)$ is commutative.

Proof

We apply Gelfand's Trick with the map $\varphi\colon G \to G$ given by $\varphi(g) = g^t$. The *p-adic Cartan decomposition* tells us all K-double cosets have a diagonal representative. Then φ is constant on this representative, so $K\varphi(x)K = KxK$.

Further research: the Iwahori–Hecke algebra

Now consider $G = \mathrm{GL}_n(\mathcal{O})$. We may quotient $\mathcal{O}$ by its unique maximal ideal $\mathcal{P}$. Then $\mathcal{O}/\mathcal{P} \cong \mathbb{F}_q$.

Then there is a map

$$\phi\colon \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathcal{P}) \cong \mathrm{GL}_n(\mathbb{F}_q)$$

given by

$$(g_{ij})_{i,j=1,\dots,n} \longmapsto (g_{ij} + \mathcal{P})_{i,j=1,\dots,n}.$$

Then the Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O})$ is $I := \phi^{-1}(B(\mathbb{F}_q))$.

The Iwahori–Hecke algebra is the Hecke algebra $C_c(I \backslash G / I)$.

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