UQ MATHS STRADBROKE ISLAND WORKSHOP ON CHARACTER VARIETIES

LECTURES BY MASOUD KAMGARPOUR

CONTENTS

LECTURE 1 - SUNDAY

Throughout today, let G be a finite group, and let Γ be a finitely presented group. Consider the group of homomorphisms $\Gamma \to G$, denoted $\operatorname{Hom}(\Gamma, G)$. We wish to count $|\operatorname{Hom}(\Gamma, G)|$. Frobenius recognised that we can phrase this question in representation-theoretic language.

Exercise 1. *Show that $|\operatorname{Hom}(\mathbb{Z}\times\mathbb{Z}, G)| = |G|\cdot c_G$, where c_G is the number of conjugacy classes of G.*

Write $\Gamma = \langle x_1,\dots,x_n \mid w_1,\dots,w_m\rangle$, where the x_i are generators and the w_i are relations. Let us first assume that Γ is a one-relator group, i.e. $m = 1$. Examples include $\mathbb{Z}/l\mathbb{Z} = \langle x \mid x^l = 1\rangle$, $\mathbb{Z}\times\mathbb{Z} = \langle x, y \mid xyx^{-1}y^{-1} = 1\rangle$ and $\langle x, y, z \mid x^3y^2x^{-1}z^5y^{10} = 1\rangle$. We also have the examples $\langle x, y \mid x = 1\rangle = \mathbb{Z}$ and $\langle x_1,\dots,x_{n+1} \mid x_{n+1} = 1\rangle = F_n$, the free group on n variables.

When Γ is the fundamental group of a surface (i.e. a two-dimensional manifold, either smooth or topological), then Γ is (usually) a one-relator group.

Exercise 2. *Let Σ be a two-dimensional manifold. Determine $\pi_1(\Sigma)$, the fundamental group of Σ.*

When Γ is a one-relator group, then a homomorphism $f\colon \Gamma \to G$ is just an assignment $x_1 \mapsto g_1, x_2 \mapsto g_2, \dots, x_n \mapsto g_n$ such that $w(g_1,\dots,g_n) = 1$. Thus, $\operatorname{Hom}(\Gamma, G) = \{(g_1,\dots,g_n) \in G^n \mid w(g_1,\dots,g_n) = 1\}$. So $|\operatorname{Hom}(\Gamma, G)| = \{(g_1,\dots,g_n) \in G^n \mid w(g_1,\dots,g_n) = 1\}|$, meaning we simply want to count the number of solutions in G^n to the equation $w(g_1,\dots,g_n) = 1$.

Definition 3. *Define $f^\Gamma = f^w\colon G \to \mathbb{Z} \hookrightarrow \mathbb{C}$ by $g \mapsto |\{(g_1,\dots,g_n) \in G^n \mid w(g_1,\dots,g_n) = g\}|$.*

Then we simply wish to determine $f^w(1)$.

Claim 4. *The map f^w is conjugation invariant. That is, $f^w(hgh^{-1}) = f^w(g)$ for all $g, h \in G$.*

Proof. Notice that

$$w(g_1,\dots,g_n) = g \iff w(hg_1h^{-1},\dots,hg_nh^{-1}) = hgh^{-1}.$$

Thus, we have a bijection

$$\{(g_1,\dots,g_n) \in G^n \mid w(g_1,\dots,g_n) = g\} \longleftrightarrow \{(g_1,\dots,g_n) \in G^n \mid w(g_1,\dots,g_n) = hgh^{-1}\}. \qquad \square$$

So $f^w \in \mathbb{C}[G]^G$, the space of class functions $G \to \mathbb{C}$. The fundamental theorem of representation theory states that irreducible characters of G form an orthonormal basis of $\mathbb{C}[G]^G$. Here, the orthonormal basis is with respect to the inner product

$$\langle \varphi, \psi \rangle := \int_G \varphi(x)\overline{\psi(x)}\, \mathrm{d}x = \frac{1}{|G|} \sum_{x \in G} \varphi(x)\overline{\psi(x)}.$$

Then, given any f^w, one may write it as a linear combination of basis vectors

$$f^w = \sum_{\chi \in \hat{G}} a_\chi^w \chi,$$

where $\hat{G}$ denotes the set of irreducible characters of G. For any given f^w, the coefficients are explictly given by $a_\chi^w = \langle f^w, \chi \rangle$. Futhermore, one has a second formulation of the coeffecients:

Exercise 5. *Show that*

$$a_\chi^w = |G|^{n-1} \int_{G^n} \overline{\chi(w(\underline{g}))}\, \mathrm{d}\underline{g}.$$

Frobenius considered the case $\Gamma = \mathbb{Z}/2\mathbb{Z} = \langle x \mid x^2 = 1 \rangle$ (where G is still arbitrary). Thus

$$a_\chi = a_\chi^w = \int_G \chi(w(g))\, \mathrm{d}g = \int_G \chi(g^2)\, \mathrm{d}g.$$

Exercise 6. *Let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be the n-dimensional complex representation of G whose character is χ. Then, in fact, a_χ is also given by*

$$a_\chi = \begin{cases} 1, & \text{if } \rho \text{ can be realised over } \mathbb{R} \text{ (up to isomorphism)}, \\ -1, & \text{if } \chi \text{ is real-valued but } \rho \text{ cannot be realised over } \mathbb{R} \text{ (up to isomorphism)}, \\ 0, & \text{if } \chi(G) \not\subseteq \mathbb{R} \text{ (i.e. } \chi \text{ is not always real-valued)}. \end{cases}$$

This is called the Frobenius–Schur indicator.

Remark 7. *The first case is called the real case, the second case is called the quaternionic case, and the last case is called the complex case. Why are quaternions appearing here? What justifies this name?*

Exercise 8. *Let $w = xyx^{-1}y^{-1}$ and $\Gamma = \mathbb{Z} \times \mathbb{Z} = \langle x, y \mid w \rangle$. Then we know that*

$$a_\chi^w = \int_{G^2} \chi(xyx^{-1}y^{-1})\, \mathrm{d}x\, \mathrm{d}y.$$

Show that $a_\chi^w = \frac{1}{\chi(1)}$.

LECTURE 2 - MONDAY

Last time, we considered the problem of counting $|\operatorname{Hom}(\Gamma, G)|$ for finite G and finitely presented Γ. Then

$$a_\chi^w = c\int_{G^n} \overline{\chi(w(x))}\,\mathrm{d}x \in \mathbb{C},$$

for some normalisation constant $c = c(G, n)$ depending on G, n and the choice of Haar measure. For $w(x) = x^2$, we have $a_\chi^w \in \{0, -1, 1\}$. In the literature, for this choice of w, the value a_χ^w is usually denoted by $\nu_2(\chi)$, the Frobenius–Schur indicator. If G is a reductive group over $\mathbb{F}_q$ and $\chi \in \widehat{G(\mathbb{F}_q)}$, then $\nu_2(\chi)$ is not known in general.

Exercise 9. *Use Exercise 8 to rederive Exercise 1.*

Remark 10. *Let*

$$G = \mathrm{UL}_n(\mathbb{F}_q) := \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \ddots & \vdots \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we do not know in general what $c_{\mathrm{UL}_n(\mathbb{F}_q)}$ is. We do not even know if this number is a polynomial in q. This is relevant to a famous conjecture of Higman.

Exercise 11. *Compute $c_{\mathrm{GL}_2(\mathbb{F}_q)}$, $c_{\mathrm{GL}_3(\mathbb{F}_q)}$ and $c_{\mathrm{GL}_n(\mathbb{F}_q)}$.*

Suppose w_1 and w_2 share no alphabets. That is, w_1 and w_2 do not share variables (e.g. $\langle x, y, z \mid x^2y = z^3 = 1\rangle$, where $w_1(x,y,z) = x^2y$ and $w_2(x,y,z) = z^3$ do not share variables). We may also call w_1 and w_2 disjoint. We write $w_1 \star w_2$ to denote the word formed by concatenating w_1 and w_2 (e.g. $(xy)\star(x^{-1}y^{-1}) = xyx^{-1}y^{-1}$).

Our goal is to show that, if we know the functions f^{w_1} and f^{w_2}, where w_1 and w_2 are disjoint, then we can determine $f^{w_1\star w_2}$. Recall that $f^w\colon G \to \mathbb{Z} \hookrightarrow \mathbb{C}$ is given by $g \mapsto |\{\underline{g} \in \mathbb{G}^n \mid w(\underline{g}) = g\}|$.

Exercise 12. *Show that the complex group algebra $\mathbb{C}[G]$ is isomorphic as an algebra to the algebra of complex-valued functions on G, denoted $\mathrm{Fun}(G) := \{$functions $f\colon G \to \mathbb{C}\}$. Also show that these algebras are commutative if and only if G is.*

Inside the space $\mathbb{C}[G] \cong \mathrm{Fun}(G)$ we have the notion of convolution, given by

$$(\varphi \star \psi)(g) := \sum_{\substack{x,y\in G\\ xy=g}} \varphi(x)\psi(y) = \sum_{h\in G}\varphi(h)\psi(h^{-1}g) = \int_G \varphi(h)\psi(h^{-1}g)\,\mathrm{d}h.$$

Exercise 13. *Let χ and χ' be irreducible characters of G. Then show that*

$$\chi \star \chi' = \begin{cases} 0, & \text{if } \chi \neq \chi', \\ \frac{|G|}{\chi(1)}\chi, & \text{if } \chi = \chi' \text{ (normalisation constant may be different, TBD).} \end{cases}$$

Exercise 14. *If w_1 and w_2 are disjoint, then*

$$f^{w_1\star w_2} = f^{w_1} \star f^{w_2}.$$

Exercise 15. *Use the previous exercise to compute a_χ^w and $f^w(1)$ when* *(i)* $w(x_1, \ldots, x_n, y_1, \ldots, y_n) := [x_1, y_1] \cdot \ldots \cdot [x_n, y_n]$,

(ii) $w(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) := [x_1, y_1] \cdot \ldots \cdot [x_n, y_n] z_1 \cdot \ldots \cdot z_k$, *and*

(iii) $w(x_1, \ldots, x_n) := x_1^2 \cdot \ldots \cdot x_n^2$.

Remark 16. *The case* (i) *is the fundamental group of a compact orientable surface of genus* n*, the case* (ii) *is the fundamental group of a compact orientable surface of genus* n *with* k *punctures, and the case* (iii) *is the fundamental group of a compact non-orientable surface of genus* n.

Exercise 17. *Show that* $\pi_1(\Sigma_n)$ *is not a free group, where* Σ_n *is the compact surface of genus* n *(e.g. the torus of genus* $n \geq 1$*).*

Given a group G and a field k, we have two spaces: the space of k-valued functions on G, $\mathrm{Fun}(G) := \{f\colon G \to k\}$, and the group algebra of G over k, $k[G] := \mathrm{Span}_k\{e_g \mid g \in G\} = \{\sum_{g\in G} a_g e_g \mid a_g \in k,\ \text{sum is finite}\}$. We have a canonical pairing

$$\varphi\colon\ \mathrm{Fun}(G) \times k[G] \to k, \quad (f, \sum a_g e_g) \mapsto \sum a_g f(g).$$

Exercise 18. *Show that the pairing φ is perfect.*

This induces a canonical isomorphism $k[G]^* \simeq \mathrm{Fun}(G)$. However, when G is infinite, then $k[G]$ is an infinite dimensional space, so $k[G] \not\simeq \mathrm{Fun}(G)$.

Example 19. *Consider $G = (\mathbb{Z}, +)$ and $\mathbb{C}[\mathbb{Z}]$. This has a countable basis $\{e_n\}_{n\in\mathbb{Z}}$, but $\mathrm{Fun}(\mathbb{Z}) = \{f\colon \mathbb{Z} \to \mathbb{C}\}$ does not have a countable basis.*

If G is finite, then we are able to identify $k[G]$ with $\mathrm{Fun}(G)$ using the map $e_g \mapsto \delta_g$, where δ_g is the indicator function of g. This identification respects the multiplication of both algebras, since $\delta_g \star \delta_h = \delta_{gh}$, so $\mathrm{Fun}(G) \simeq k[G]$ as algebras when G is finite.

Now let $\chi \in \mathrm{Fun}(G)^G \simeq \mathbb{C}[G]^G = Z(\mathbb{C}[G])$. If A is an algebra, then an idempotent in A is an element satisfying $a^2 = a$. A central idempotent is an idempotent in $Z(A)$, the center of A. Then we have the following diagram:

$$\begin{tikzcd}
\mathrm{Fun}(G)^G \arrow[rr, leftrightarrow] & & \mathbb{C}[G]^G = Z(G) \\
\{\text{irreducible characters}\} \arrow[u, hook] \arrow[rr, leftrightarrow, dashed] & & \left\{\begin{array}{c}\text{central}\\ \text{idempotents in } \mathbb{C}[G]\\ \text{(up to rescaling)}\end{array}\right\}.
\end{tikzcd}$$

We present some directions for further research:

(i) Higher Frobenius–Schur indicators. Consider $\langle x \mid x^n = 1\rangle = \mathbb{Z}/n\mathbb{Z}$ and $\nu_n(\chi) = \int_G \overline{\chi(g^n)}\,\mathrm{d}g$. Does the image of this map lie in $\mu_n \cup \{0\}$? What does ν_n say about χ?

(ii) We have only talked about one-relator groups. What about two-relator groups? i.e. $\langle x_1, \dots, x_n \mid w_1, w_2\rangle$. For example, $\langle x, y, z \mid x^3yx^{-1}z = xyz = 1\rangle$. Note that $\{1\} = \langle x \mid x^3 = x^5 = 1\rangle \neq \langle x \mid x^8 = 1\rangle = \mathbb{Z}/8\mathbb{Z}$ so this problem is harder than simply concatenating words together.
Define $f^\Gamma\colon G \to \mathbb{Z}$ by $f^\Gamma(h) := |\{(g_1, \dots, g_n) \in G^n \mid w_1(g_1, \dots, g_n) = h,\ w_2(g_1, \dots, g_n) = h\}|$. We want to compute $f^\Gamma(1)$. Alternatively, we could have defined $f^\Gamma\colon G \times G \to \mathbb{Z}$ by $f^\Gamma(h, h') := |\{(g_1, \dots, g_n) \in G^n \mid w_1(g_1, \dots, g_n) = h,\ w_2(g_1, \dots, g_n) = h\}|$. Then we want $f^\Gamma(1, 1)$. There are even more ways to rephrase this question. It is not clear which way is the best.

(iii) Can we do things integrally?

(iv) What if Γ is not finitely generated or finitely presented?

Lecture 4 - Tuesday

Definition 20. *A category $\mathcal{C}$ is a pair of 'sets' $(\mathcal{O}, \mathcal{M})$, where $\mathcal{O}$ are called the objects of $\mathcal{C}$ and $\mathcal{M}$ are called the morphisms of $\mathcal{C}$, satisfying the following conditions:*

(i) For every object $x \in \mathcal{O}$, there is an associated morphism $\mathrm{Id}_x \colon x \to x$ lying in $\mathcal{M}$, called the identity on x, defined by $\mathrm{Id}_x(x') := x'$ for all $x' \in x$.

(ii) Given two morphisms $f \colon x \to y$ and $g \colon y \to z$, $\mathcal{M}$ contains a morphism denoted $g \circ f \colon x \to z$, called the composition of f and g. That is, we have the diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[rr, bend right, "g\circ f"'] & y \arrow[r, bend left, "g"] & z \end{tikzcd}$$

(iii) Composition of morphisms is associative. That is, $(g \circ f) \circ h = g \circ (f \circ h)$ for all $f \colon x \to y$, $g \colon y \to z$ and $h \colon z \to x$. Whenever we write a compostition, say $g \circ f$, we assume that the $\operatorname{dom} g$ contains $\operatorname{im} f$.

(iv) For all $f \colon x \to y$, there holds $f \circ \mathrm{Id}_x = \mathrm{Id}_y \circ f = f$.

Exercise 21. *Show that a category with one object is the same thing as a monoid.*

Example 22. *(i) $\mathsf{fSet} \subset \mathsf{Set}$, the category of finite sets and the category of sets. Specifically, Set has the collection of objects $\mathcal{O} = \{X \mid X \text{ is a set}\}$ and the collection of morphisms $\mathcal{M} = \{f \mid f \text{ is a function of sets}\}$.*

(ii) $\mathsf{AbGrp} \subset \mathsf{Grp}$, the category of abelian groups and the category of groups. Specifically, Grp has the collection of objects $\mathcal{O} = \{G \mid G \text{ is a group}\}$ and the collection of morphisms $\mathcal{M} = \{f \mid f \text{ is a homomorphism of groups}\}$.

(iii) k-$\mathsf{Mod} \subset \mathsf{Vect}_k$, the category of k-modules and the category of vector spaces over k. Specifically, Vect_k has the collection of objects $\mathcal{O} = \{V \mid V \text{ is a vector space over } k\}$ and the collection of morphisms $\mathcal{M} = \{L \mid L \text{ is a linear map of vector spaces}\}$.

(iv) $\mathsf{Man} \subset \mathsf{Top}$, the category of manifolds and the category of topological spaces. Specifically, Top has the collection of objects $\mathcal{O} = \{(X, \tau) \mid (X, \tau) \text{ is a topological space}\}$ and the collection of morphisms $\mathcal{M} = \{f \mid f \text{ is a continuous map}\}$.

(v) $\mathsf{ProjVar} \subset \mathsf{AlgVar}$, the category of projective varieties and the category of algebraic varieties. Specifically, AlgVar has the collection of objects $\mathcal{O} = \{X \mid X \text{ is an algebraic variety}\}$ and the collection of morphisms $\mathcal{M} = \{f \mid f \text{ is a morphism of varieties}\}$.

Definition 23. *Let $\mathcal{C}$ be a category. Consider the diagram*

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \end{tikzcd}$$

An inverse of f is a map $g \colon y \to x$ such that $f \circ g = \mathrm{Id}_y$ and $g \circ f = \mathrm{Id}_x$. If f has an inverse, we call f an isomorphism, and denote g by f^{-1}.

Exercise 24. *Show that the inverse, if it exists, is unique.*

Definition 25. *A groupoid is a category $\mathcal{C}$ where every morphism is an isomorphism. A groupoid with one object is called a group. Given a groupoid $\mathcal{C}$, define $\pi_0(\mathcal{C}) := \{\text{isomorphism classes of objects of } \mathcal{C}\}$ (this is a 'set') and $\pi_1(\mathcal{C}, x) := \mathrm{Aut}_{\mathcal{C}}(x)$.*

Definition 26 (The fundamental groupoid)**.** *Given a topological space X, we define a groupoid $\Pi(X)$ as follows. The objects of $\Pi(X)$ are the points in X (i.e. $\mathcal{O} = X$), and for each pair of points $(x, y) \in X^2$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is the collection of homotopy classes of paths from x to y. That is, $\mathrm{Hom}_{\mathcal{C}}(x, y) := \{[f] \mid f$ is a path from x to $y\}$. Then $\mathcal{M} = \cup_{x,y \in X} \mathrm{Hom}_{\mathcal{C}}(x, y)$.*

Recall that a path from x to y is a continuous function $f\colon [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Also recall that two paths f and g from x to y are homotopically equivalent if there exists a continuous function $F\colon [0, 1] \times [0, 1] \to X$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$.

Exercise 27. *(i) Check that $\Pi(X)$ is a groupoid.*
(ii) Show that $\pi_0(\Pi(X)) = \pi_0(X)$.
(iii) Show that $\pi_1(\Pi(X), x) = \pi_1(X, x)$.

Definition 28 (The equivalence groupoid)**.** *Let X be a set with an equivalence relation $\sim$. Then $X/\sim$ is the set of equivalence classes of X with respect to $\sim$. The equivalence groupoid $[X/\sim]$ is defined as follows. The objects of $[X/\sim]$ are the elements of X, and there is a unique morphism (i.e. isomorphism) from x to y if and only if $x \sim y$.*

Exercise 29. *Check that $[X/\sim]$ is a groupoid. Determine $\pi_0([X/\sim])$ and $\pi_1([X/\sim], x)$ for $x \in X$.*

Definition 30 (The quotient groupoid)**.** *Let G be a group acting on a set X. Then X/G is the set of orbits of X with respect to G. The quotient groupoid $[X/G]$ is defined as follows. The objects of $[X/G]$ are the elements of X, and $\mathrm{Hom}(x, y) = \{g \in G \mid g \cdot x = y\}$.*

Exercise 31. *(i) Check that $[X/G]$ is a groupoid.*
(ii) Show that $\pi_0(X/G) = X/G$.
(iii) Show that $\pi_1([X/G], x) = \mathrm{Aut}_{[X/G]}(x) = Gx =: \mathrm{stab}_G(x)$, the stabilser of x with respect to G.

Definition 32. *If $\mathcal{C}$ is a groupoid then the size of $\mathcal{C}$ is defined by*

$$|\mathcal{C}| := \sum_{[x] \in \pi_0(\mathcal{C})} \frac{1}{|\mathrm{Aut}_{\mathcal{C}}(x)|},$$

if this value exists.

Exercise 33. *Show that if X and G are finite, then $|[X/G]| = |X|/|G|$.*

Exercise 34. *Let G act on $X = G$ by conjugation. Determine the quotient groupoid $[G/G]$.*

Lecture 5 - Wednesday

Today we will discuss geometric invariant theory (GIT). Let G be a linear algebraic group over a field k, so it is an algebraic subgroup of $\mathrm{GL}_n(k)$. Also let X be a variety over k. We want to somehow make sense of the quotient of X by G *in the category of varieties*.

We will assume that X is affine, i.e. $X = \operatorname{Spec} A$, for some k-algebra A. That is, $A = k[X]$, the space of functions on X. Then we may define a certain quotient of X by G.

Definition 35. *The GIT quotient is defined by*

$$X//G := \operatorname{Spec}(A^G) = \operatorname{Spec}(k[X]^G),$$

where A^G is the G-invariant elements of A.

Remark 36. *A^G is not necessarily finitely generated (Hilbert's 14th problem). Nagata provided the first example of this by considering an action of $\mathbb{C}^3$ on $\mathbb{C}^{48}$.*

Theorem 37 (Hilbert)**.** *If G is a reductive group over k then $A^G = k[G]^G$ is finitely generated.*

Example 38. *Now let $G = \mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = 1\rangle$. Then consider the action of G on $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$ given by $(a \cdot p)(x) := p(-x)$. The orbits of this action are $\{\{x, -x\} \mid x \neq 0\} \sqcup \{0\}$. If $x \neq 0$ then $G_x := \mathrm{stab}_G(x) = \{1\}$, and if $x = 0$ then $G_x = G$. Now $k[x]^{S_2} = \{p(x) \in k[x] \mid p(x) = p(-x)\} = k[x^2] = k[z]$, where we relabel $z = x^2$. Then $\operatorname{Spec} k[x]^{S_2} = \operatorname{Spec} k[z] = \mathbb{A}^1$.*

What are the points of $X//G$? Can we relate them to the orbits? We have the map $k[G]^G \hookrightarrow k[X]$ which induces the diagram

$$\begin{tikzcd} \operatorname{Spec} k[X] \arrow[r] \arrow[d, equal] & \operatorname{Spec} k[X]^G \arrow[d, equal] \\ X \arrow[r, "\pi"] & X//G \end{tikzcd}$$

Is π surjective? The map $\pi\colon \mathbb{A}^1 = \operatorname{Spec} k[x] \to \mathbb{A}^1 = \operatorname{Spec} k[x^2] = \operatorname{Spec} k[z]$ is given by $\alpha \mapsto \alpha^2$. This is a surjective morphism of algebraic varieties.

We consider the fibres of π. These are given by $\pi^{-1}(z) = \{$square roots of $z\} = \{0\}$ or $\{\sqrt{z}, -\sqrt{z}\}$. So in this case, $X//G$ is just the orbit space X/G.

Example 39. *Consider the action of $G = S_2$ on $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ given by $(\sigma \cdot p)(x, y) := p(y, x)$. Then $k[x, y]^{S_2}$ is the k-algebra of symmetric polynomials in two variables. Let us write down symmetric polynomials for each degree. For degree 1, the only symmetric polynomial is $x+y$. For degree 2, we have xy, x^2+y^2, x^2+y^2-10xy, $(x+y)^2$, and so on. For degree 3, we have x^3+y^3, $x^3+y^3+x^2+y^2$, and so on. Call $e_1 = x+y$ and $e_2 = xy$.*

Exercise 40. *(i) Let k be any ring. Show that the algebra $k[x, y]^{S_2}$ is generated by e_1 and e_2. That is, show that*

$$k[x, y]^{S_2} = k[e_1, e_2] = k[\alpha, \beta],$$

where α and β are new variables.

(ii) Let k be a field of characteristic 0. Show that $k[x, y]^{S_2} = k[x+y, x^2+y^2] = k[\alpha, \beta]$. Note that this doesn't work for every ring. Show that it is not true for the ring $\mathbb{Z}$. Is it true for the field $\mathbb{F}_p$?

Example 39 (continued). *Now we have that* $\mathbb{A}^2//S_2 = \operatorname{Spec} k[x,y]^{S_2} = \operatorname{Spec} k[x,y] = \mathbb{A}^2$. *We have the map* $\operatorname{Spec} k[x,y]^{S_2} = k[x+y, xy] \hookrightarrow k[x,y]$. *We have the diagram*

$$\begin{tikzcd} \operatorname{Spec} k[x,y] \arrow[r, two heads, "\pi"] \arrow[d, equal] & \operatorname{Spec} k[x,y]^{S_2} \arrow[d, equal] \\ \mathbb{A}^2 \arrow[r, two heads] & \mathbb{A}^2 \end{tikzcd}$$

$$(x,y) \longmapsto (x+y, xy)$$

Exercise 41. *Compare the fibres of* π *with the orbits of* X *with respect to the action of* G *(i.e.* X/G*).*

Exercise 42. *Let* $G = S_2$ *act on* $X = \mathbb{C}^\times \times \mathbb{C}^\times$ *by* $\sigma \cdot (x,y) := (y,x)$. *Compute* $X//G$ *and describe the fibres of* $\pi \colon X \to X//G$.

Exercise 43. *Let* $G = S_2$ *act on* $X = \mathbb{C}^2$ *by* $\sigma \cdot (x,y) := (-x,-y)$. *Compute* $X//G$. *Is it singular? That is, is it a smooth manifold? Here, smooth means infinitely differentiable.*

Lecture 6 - Thursday

Last time we considered a linear algebraic group G acting on an affine variety X. We defined the GIT quotient $X//G := k[X]^G$, the G-invariant k-valued functions on X. Then we had the diagram

$$\begin{tikzcd}
k[X]^G \arrow[r, hook] \arrow[d, squiggly] & k[X] \\
X = \operatorname{Spec} k[X] \arrow[r, two heads, "\pi"'] & X//G := \operatorname{Spec} k[X]^G
\end{tikzcd}$$

Example 44. *Let $G = \mathrm{GL}_2(\mathbb{C})$ act on $X = \mathfrak{g} = \mathrm{Mat}_2(\mathbb{C})$ by conjugation. Our goal is to determine $\mathfrak{g}//G$, which is called the adjoint quotient. Then $f \in \mathbb{C}[\mathfrak{g}]^G$ if and only if $f(gyg^{-1} = f(y)$ for all $g \in G$ and $y \in \mathfrak{g}$. Then f is constant on conjugacy classes.*

By the Jordan normal form, f is completely determined by its value on the list of conjugacy class represenatives

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda, \mu \in \mathbb{C}.$$

We claim that $f(\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)) = f(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right))$. To see this, observe that

$$f\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = f\left(\lim_{t \to 0^+} \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}\right) = \lim_{t \to 0^+} f\left(\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}\right) = \lim_{t \to 0^+} f\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = f\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right).$$

Then we do not need to consider the conjugacy class represented by $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. Thus, we have an inclusion

$$\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\lambda, \mu]_2^S = \mathbb{C}[\lambda + \mu, \lambda\mu] = \mathbb{C}[\lambda + \mu, \lambda^2 + \mu^2].$$

This map is also surjective. To see this, consider the maps $\mathrm{tr}, \det \in \mathbb{C}[\mathfrak{g}]^G$. We have a map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\lambda, \mu]$ which maps $\mathrm{tr} \mapsto \lambda + \mu$ and $\det \mapsto \lambda\mu$. Then we have the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, two heads] \arrow[dr, two heads] & \mathfrak{g}//G = \mathbb{C}^2/S_2 \arrow[d, "\wr"] \\
& \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, mapsto] \arrow[dr, mapsto] & \{\lambda, \mu\} \text{ eigenvalues of } A \arrow[d, mapsto] \\
& (\mathrm{tr}\, A, \det A) = (\lambda + \mu, \lambda\mu)
\end{tikzcd}$$

Example 45. *Now consider $G = \mathrm{GL}_n(\mathbb{C})$ acting on $\mathfrak{g} = \mathrm{Mat}_n(\mathbb{C})$ by conjugation. Let $\mathfrak{h} = \{$diagonal matrices in $\mathfrak{g}\}$. Then we have the diagram*

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, hook] & \mathfrak{g} \arrow[d, squiggly] & \\
& \mathbb{C}[\mathfrak{g}] \arrow[r, two heads] & \mathbb{C}[\mathfrak{h}] \\
& \mathbb{C}[\mathfrak{g}]^G \arrow[u, hook] \arrow[ur] \arrow[r, "r"'] & \mathbb{C}[\mathfrak{h}]^W \arrow[u, hook]
\end{tikzcd}$$

We claim that r is an isomorphism. To see that r is injective, suppose that $f \in \mathbb{C}[\mathfrak{g}]^G$. Then $f(x) = f(gxg^{-1}) = 0$ for all $x \in \mathfrak{h}$ and $g \in G$. Then $f(y) = 0$ for all $y \in \mathfrak{g}$ which are diagonalisable (i.e. semisimple), denoted $\mathfrak{g}^{ss}$. We claim that $\mathfrak{g}^{ss}$ is Zariski-dense in $\mathfrak{g}$ (i.e. if $f \in \mathbb{C}[\mathfrak{g}]$ and $f(\mathfrak{g}^{ss}) = \{0\}$ then $f(\mathfrak{g}) = \{0\}$). This shows injectivity.

For surjectivity, we need the following claim. We claim that $\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n]$ where $e_1 = \sum x_i$, $e_2 = \sum x_i x_j$, ..., $e_n = \sum \prod x_i$. Then surjectivity follows by taking coefficients of characteristic polymomials.

Now we have the map

$$\pi\colon\ \mathfrak{g} \to \mathfrak{g}//G \simeq \mathfrak{h}//W, \quad A \mapsto \textit{multiset of eigenvalues of } A.$$

Then $\pi^{-1}(S) = \{$matrices whose multiset of eigenvalues is $S\}$, and

$$\pi^{-1}(\{\lambda,\lambda,\mu\}) = \left\{ \textit{all matrices conjugate to } \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix} \textit{ or } \begin{pmatrix} \lambda & 1 & \\ & \lambda & \\ & & \mu \end{pmatrix} \right\}.$$

There is a unique semisimple (i.e. closed) orbit in each fibre.