The Twin Prime and Elliot-Halberstam Conjectures

Bailey Whitbread

October 9, 2020

Last Time

- Sieve of Eratosthenes:

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

- Idea of a Sieve:

  Finite set $\mathcal{A} \subset \mathbb{Z}$,
  a set of primes $\mathcal{P}$,
  an integer $z > 1$.

- We estimate

  $$S(\mathcal{A}, \mathcal{P}, z) = |\{a \in \mathcal{A} : p \nmid a, p \in \mathcal{P}, p < z\}|.$$

- We used the Mobius function μ to derive the estimate

  $$\pi(N) < \frac{N}{\log \log N} + \mathcal{O}(N^{\log 2}).$$

Twin Prime Conjecture

Define p_n to be the n^{th} prime number.
The expression $p_{n+1} - p_n$ is the n^{th} prime gap.

$$3 - 2 = 1,\ 5 - 3 = 2,\ 7 - 5 = 2,\ 11 - 7 = 4,\ 13 - 11 = 2,$$

$$17 - 13 = 4,\ 19 - 17 = 2,\ 23 - 19 = 4,\ 29 - 23 = 6,\ \ldots$$

Twin Prime Conjecture

Is $p_{n+1} - p_n = 2$ infinitely often?

ie. Is the following true:

$$\liminf_{n \to \infty} p_{n+1} - p_n = 2$$

Elliot-Halberstam Conjecture

Elliot-Halberstam Conjecture

For “most” $q \in \mathbb{N}$, there exists a “good” Prime Number Theorem for arithmetic progressions in all primitive classes modulo q.

Twin Prime Conjecture

- $\liminf_{n\to\infty} p_{n+1} - p_n \leq 7 \times 10^7$. (Yitang Zhang, May 2013).
- $\liminf_{n\to\infty} p_{n+1} - p_n \leq 4680$. (Polymath Project, July 2013).
- $\liminf_{n\to\infty} p_{n+1} - p_n \leq 600$. (James Maynard, Nov 2013).
- $\liminf_{n\to\infty} p_{n+1} - p_n \leq 246$. (Polymath, April 2014).

PNT Variants

Prime Number Theorem

$\frac{x}{\log x}$ is a "good" approximation for $\pi(x)$, written

$$\pi(x) \sim \frac{x}{\log x}.$$

Prime Number Theorem for Arithmetic Progressions

If $(a, q) = 1$ then write $\pi(x; q, a)$ to mean the number of primes less than x that are congruent to a modulo q. Then

$$\pi(x; q, a) \sim \frac{1}{\phi(q)} \frac{x}{\log x}.$$

von Mangoldt Function

Define the *von Mangoldt* function $\Lambda : \mathbb{N} \to [0, \infty)$ by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some } k \geq 1, \\ 0, & \text{else.} \end{cases}$$

$$eg. \quad \Lambda(9) = \log 3, \quad \Lambda(128) = \log 2, \quad \Lambda(6) = 0.$$

It has the useful condition that

$$(\Lambda \star 1)(n) = \sum_{d|n} \Lambda(d) = \log n \implies \Lambda(n) = \sum_{d|n} \mu(d) \log(n/d).$$

Then $\Lambda = \mu \star \log$. Also we observe the inequality

$$\frac{(1-\varepsilon)(\pi(x) + \mathcal{O}(x^{1-\varepsilon}))}{\frac{x}{\log x}} \leq \frac{1}{x} \sum_{n \leq x} \Lambda(n) \leq \frac{\pi(x)}{\frac{x}{\log x}}.$$

PNT Variants

PNT (von Mangoldt function)

$$\sum_{n \le x} \Lambda(n) \sim x.$$

PNT for Arithmetic Progressions (von Mangoldt function)

If $(a, q) = 1$ then

$$\sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) \sim \frac{1}{\phi(q)} \sum_{n \le x} \Lambda(n).$$

Elliot-Halberstam Conjecture

Elliot-Halberstam at level θ

Fix $0 < \theta < 1$. We say the Elliot-Halberstam conjecture holds at level θ, written EH[θ], if we take the error in the PNT for Arith. Prog. ...

$$\left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \le x} \Lambda(n) \right|$$

... then we look at the worst error ...

$$\sup_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \le x} \Lambda(n) \right|$$

Elliot-Halberstam Conjecture

Elliot-Halberstam at level θ

$$\sup_{a\in(\mathbb{Z}/q\mathbb{Z})^\times}\left|\sum_{\substack{n\leq x\\ n\equiv a \mathrm{mod} q}}\Lambda(n)-\frac{1}{\phi(q)}\sum_{n\leq x}\Lambda(n)\right|$$

... then we sum over all q up to a certain level ...

$$\sum_{q\leq x^\theta}\sup_{a\in(\mathbb{Z}/q\mathbb{Z})^\times}\left|\sum_{\substack{n\leq x\\ n\equiv a \mathrm{mod} q}}\Lambda(n)-\frac{1}{\phi(q)}\sum_{n\leq x}\Lambda(n)\right| =: (\star)$$

... and then we say EH[θ] if this quantity above satisfies

$$(\star) << x\log^{-A}x=\frac{x}{\log^A x}$$

for all $A>0$.

Elliot-Halberstam Conjecture

What is known:

- EH[θ] is true for $0 < \theta < \frac{1}{2}$ (Bombieri-Vinogradov).
- If EH[θ] is true for all $0 < \theta < 1$, then $\liminf_{n\to\infty} p_{n+1} - p_n \leq 12$.
- If the *generalized* EH conjecture is true, then $\liminf_{n\to\infty} p_{n+1} - p_n \leq 6$.

Generalized Elliot-Halberstam Conjecture

Recall that $\Lambda = \mu \star \log$ and EH[θ] is concerned with the quantity

$$\sum_{q \le x^\theta} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \le x} \Lambda(n) \right|$$

If we replace Λ with some other pair of "nice" arithmetic functions α and β then the generalised Elliot-Halberstam conjecture is concerned with the quantity

$$\sum_{q \le x^\theta} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} (\alpha \star \beta)(n) - \frac{1}{\phi(q)} \sum_{n \le x} (\alpha \star \beta)(n) \right|$$

Admissible Prime k-Tuples

Admissible Prime k-Tuples

We say that $(h_1, h_2, ..., h_k)$ where $h_1 < h_2 < ... < h_k$ and $h_i \in \mathbb{N}$ is an *admissible prime k-tuple* if the tuple "avoids" at least one congruence class modulo p for every prime p.

eg. $(0, 1)$ is not admissible because when $p = 2$, we notice $0 \in [0]_2$ and $1 \in [1]_2$ so $(0, 1)$ "fills" the congruence classes modulo 2.

eg. $(0, 2)$ is admissible because when $p = 2$, we notice $0, 2 \notin [1]_2$.

Advancements in the Twin Prime Conjecture can come from finding admissible k-tuples.